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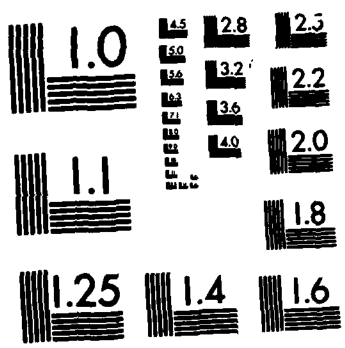


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# NONEXPLICIT SINGULAR PERTURBATIONS AND INTERCONNECTED SYSTEMS

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Singular perturbations have been shown to be an effective tool in the analysis and design of systems with "slow" and "fast" dynamics. However, the use of this tool is often inhibited by the fact that when physical quantities are selected as state variables, the model fails to be in the standard singularly perturbed form. In this thesis we deal with such nonexplicit models and show that for a wide class of problems a proper selection of variables leads to explicit singularly perturbed models. Equilibrium and conservation properties are shown to provide a coordinate-free characterization of two-time-scale

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# NONEXPLICIT SINGULAR PERTURBATIONS AND INTERCONNECTED SYSTEMS

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George Michael Peponides

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# NONEXPLICIT SINGULAR PERTURBATIONS AND INTERCONNECTED SYSTEMS

BY

GEORGE MICHAEL PEPONIDES

Dipl., University of Athens, 1976
M.S., University of Illinois, 1979

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Electrical Engineering
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Thesis Adviser: Professor P. V. Kokotovic

Urbana, Illinois

TO MY FATHER, AND IN MEMORY OF MY DEAR MOTHER

# NONEXPLICIT SINGULAR PERTURBATIONS

# AND INTERCONNECTED SYSTEMS

George Michael Peponides, Ph.D.
Department of Electrical Engineering
University of Illinois at Urbana-Champaign, 1982

Singular perturbations have been shown to be an effective tool in the analysis and design of systems with "slow" and "fast" dynamics. However, the use of this tool is often inhibited by the fact that when physical quantities are selected as state variables the model fails to be in the standard singularly perturbed form. In this thesis we deal with such nonexplicit models and show that for a wide class of problems a proper selection of variables leads to explicit singularly perturbed models. Equilibrium and conservation properties are shown to provide a coordinate-free characterization of two-time-scale systems. They also suggest a coordinate transformation that transforms nonexplicit models into explicit ones. This transformation is then used to study nonlinear high gain feedback systems, thus extending earlier linear results. It is also utilized to establish the relation between weak connections and time scales in interconnected systems whose subsystems possess a continuum of equilibrium points. Finally, the methodology is applied to reduced order modeling of dynamic networks and it is shown that linear conservation laws lead to a linear transformation separating the time scales even when some of the components of the network are nonlinear. Moreover, the reduced order model retains the physical meaning of the original system.

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# CHAPTER 1

# INTRODUCTION

## 1.1 Model Simplification in Large Systems

An issue of paramount importance in the study of large scale systems is that of model simplification or reduced order modeling. The sheer size on the one hand and the richness and complexity of phenomena on the other make the use of detailed models in the analysis and control of large systems impractical if not impossible. A good analyst or designer knows that a model should encompass only the "relevant" behavior of the system and should not be cluttered with unnecessary detail. Although this may sometimes be accomplished by employing parsimonious models for the components of the system, there are cases where further simplification is needed to make the model manageable both computationally and conceptually. A characteristic example arises in stability studies of interconnected power systems where the use of the crudest model for each generator (the so-called electromechanical model) results in hundreds or even thousands of state variables. It is thus desirable to have systematic model order reduction methods for which the approximation involved can be estimated.

Singular perturbations is a well documented [1-4] method for reduced order analysis and design, in which dynamic phenomena of widely different speeds are treated separately. In the short run the slow dynamics are essentially constant and the focus is on the fast ones. In the long run the fast dynamics settle to their "quasi-steady-state" and the focus is on the slow dynamics. This time-scale thinking is common

in diverse engineering fields [5-7]. If a small parameter $\varepsilon$ representing the speed ratio of slow and fast dynamics can be identified this intuitively appealing idea leads to asymptotic analysis. Most of the literature [8-10] is devoted to systems of the form

$$\begin{aligned} \dot{y} &= f(y,z,\varepsilon) \qquad & y(0) &= y_o \\ \varepsilon\dot{z} &= g(y,z,\varepsilon) \qquad & z(0) &= z_o \end{aligned} \tag{1.1}$$

where $\varepsilon$ multiplies the z-derivatives, y is a $\nu$-vector and z is a $\rho$-vector. Formally setting $\varepsilon=0$ in (1.1), solving

$$0 = g(\bar{y},\bar{z},0) \tag{1.2}$$

for $\bar{z}$, $\bar{z} = \Psi(\bar{y})$ and substituting into (1.1)

$$\frac{d\bar{y}}{dt} = f(\bar{y}, \Psi(\bar{y}),0) \triangleq \bar{f}(\bar{y}), \qquad \bar{y}(0) = y_o \tag{1.3}$$

we obtair he slow reduced model. Writing system (1.1) in the "stretched" time variable $\tau = \frac{t}{\varepsilon}$ and setting $\varepsilon=0$ we obtain the fast reduced system (or associated system or boundary layer system)

$$\frac{d\tilde{z}}{d\tau} = g(y_o, \bar{z}_o + \tilde{z}, 0) \qquad \tilde{z}(0) = z_o-\bar{z}_o \tag{1.4}$$

where $z_o = \bar{z}(0)$. Variables $y,z,\varepsilon,t$ are restricted to lie in a domain $D : \|y-\bar{y}(t)\| < r$, $\|z-\bar{z}(t)\| < r$, $0 \leq \varepsilon \leq \varepsilon_o$, $0 \leq t \leq T$, where $r > \|y_o-\bar{y}(0)\|$. The following theorem relates the solutions of (1.1) with the solutions of (1.3)-(1.4).

Theorem 1.1 [35] Let the following conditions be satisfied.

H1. $f$, $\partial f/\partial y$, $\partial f/\partial z$, $g$, $\partial g/\partial y$, $\partial g/\partial z$ are of class $C^0$ in D.

H2. The solution $\tilde{z}(\tau)$ of (1.4) exists on $\tau \in [0,\infty]$, is unique, and is asymptotically stable with respect to $\tilde{z}=0$.

H3. The solution $\bar{y}(t)$ of the reduced system (1.3) exists and is unique on $t \in [0,T]$.

H4. The real parts of the eigenvalues of the Jacobian matrix

$$\partial g/\partial z(\bar{y},\bar{z},0) \tag{1.5}$$

are negative on $[0,T]$, for $\bar{z}=\Psi(y)$.

Then for sufficiently small $\varepsilon$, the full system (1.1) has a unique solution $y(t,\varepsilon)$ on $t \in [0,T]$ satisfying the initial condition $y(0,\varepsilon)=y_o$, $z(0,\varepsilon)=z_o$. Furthermore,

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = \bar{y}(t) \qquad \text{on } [0,T] \tag{1.6}$$

$$\lim_{\varepsilon\to 0} z(t,\varepsilon) = \bar{z}(t)+\tilde{z}\left(\frac{t}{\varepsilon}\right) \qquad \text{on } [0,T] \tag{1.7}$$

where the limits in (1.6), (1.7) are uniform in t on [0,T].

From (1.6), (1.7) the response of y in (1.1) is approximated, to $O(\varepsilon)$, by the response of the slow system (1.3) whereas the response of z is approximated by a boundary layer $\tilde{z}(\frac{t}{\varepsilon})$ superimposed on the quasi-steady-state $\bar{z}(t)=\Psi(\bar{y}(t))$.

An extensive literature dealing with system (1.1) and the corresponding controlled system includes results on stability [20,65,66], linear [4,67] and nonlinear [3] regulator design, controllability properties [37] and time-optimal control [68], filtering and smoothing [69]. A basic

assumption in these references is that the Jacobian matrix $\partial g/\partial z(\bar{y},\bar{z},0)$ is nonsingular. When this happens we say that time scales in (1.1) are explicit, that is, they coincide with the decomposition of the state vector into y and z. When, however, $\partial g/\partial z$ is singular time scales in (1.1) are nonexplicit, that is, all states may be mixed having fast and slow parts. Some authors treat such cases as "singular-singularly perturbed" systems [11-13] or "generalized singularly perturbed" systems [14,15].

In this thesis we take an alternate route. We recognize that in a wide class of systems singularity of $\partial g/\partial z$ is due to the selection of state variables; hence a nonsingular transformation removes the singularity of $\partial g/\partial z$, and defines new states in which time-scales are explicit. This approach has two advantages. First it puts the system into a form in which the results alluded to before can be applied. Second, from the transformed system we can easily define fast and slow reduced systems describing the system behavior in the short run and in the long run. Following this approach we establish a relation between weak connections and time scales in a class of interconnected systems. Separation of time scales in such systems leads to a physical decomposition into a slow core and a number of weakly coupled fast subsystems. The results are further specialized to structured interconnected systems such as power systems and other dynamic networks.

## 1.2 Chapter Preview

Chapter 2 starts with a simple RC example pointing the relationship between equilibrium and conservation properties on the one hand and time scales on the other [16]. These properties are used in the construction of

a transformation that makes the time scales in Linear Time Invariant (LTI) systems explicit. Next a multi-time-scale system is viewed as a succession of two-time-scale ones. Starting from the fastest time scale we proceed to the slower ones, using at each step the transformation that makes time-scales explicit. This procedure defines a sequence of "nested" reduced order models. The transformation separating the time scales is then generalized to LTI systems with inputs.

In Chapter 3 the equilibrium and conservation reasoning is extended to nonlinear systems leading to a transformation that makes time scales explicit in models of the form

$$\varepsilon\dot{x} = h(x,\varepsilon). \tag{1.8}$$

It is then shown that the results of [17-19] on high gain feedback and disturbance decoupling generalize to a class of nonlinear systems for which the controls enter linearly but the output map and feedback law are nonlinear. We next turn to interconnected systems made of systems with equilibrium manifolds and show that weak connections give rise to two-time-scale behavior. A decomposition of interconnected systems into a slow core and fast local systems leads to decentralized stability criteria based on the results of [20].

Chapter 4 deals with time scales, coherency and aggregation in nonlinear dynamic networks. Coherency based aggregation [21-23], a common procedure for order reduction in power systems, is given theoretical foundations for nonlinear electromechanical models, thus extending the results of [24-29]. It is shown that linear physical laws result in linear time-scale separating transformation even when some components of

the network are nonlinear. A five-machine power system example illustrates the proposed reduced-order modeling and verifies its validity.

Extensions in several directions and possible uses of the decomposition in direct stability analysis are discussed in Chapter 5.

# CHAPTER 2

# MODELING OF TWO-TIME-SCALE SYSTEMS

## 2.1 Introduction

When the model of a real system with the two-time-scale property is expressed in terms of physical variables it often fails to be in the form (1.1). An important requirement in (1.1) is that $\partial g/\partial z$ be nonsingular along $\bar{z}(t)$. When this condition is violated the model is said to be *nonexplicit* and the conclusions in Chapter 1 have to be modified.

Some authors [11-13] treat nonexplicit models as "singular-singularly perturbed" systems. Instead we approach them from the modeling point of view recognizing that the singularity of $\partial g/\partial z$ is due to the choice of state variables. We show that equilibrium [16] and conservation properties provide a coordinate-free characterization of singular perturbations. These properties are used in the construction of a transformation leading to the explicit model (1.1) with the slow part of $z(t)$ being $O(\varepsilon)$. The discussion in this chapter is restricted to Linear Time Invariant (LTI) systems. Extension of the basic ideas to nonlinear systems and applications to interconnected systems, high gain feedback and dynamic networks appear in Chapters 3 and 4.

In Section 2.2, a simple physical system is used to motivate the discussion and indicate the relation between time scales on the one hand and equilibrium and conservation properties on the other. In Section 2.3, the relation is established for LTI systems and a transformation is constructed that transforms a nonexplicit singularly perturbed model to the explicit

model (1.1). In Section 2.4, multi-time-scale systems are treated as a succession of two-time-scale systems and a sequence of nested reduced order models is defined. Section 2.5 generalizes the results of Section 2.3 to systems with inputs and Section 2.6 deals with some structured nonexplicit models.

## 2.2 Equilibrium and Conservation Properties

Although nonexplicit singular perturbations occur in as simple systems as RC-circuits they have not attracted much attention. In contrast, explicit perturbations have been investigated for networks with "parasitic" inductances and capacitances [30,31]. When such parasitics are expressed as multiples of $\varepsilon$ and capacitor voltages and inductor currents are used as state variables, the circuit model is in the explicit form (1.1). A simple illustration is the RC-circuit of Fig. 2.1a with state equations

$$\begin{aligned}
(R_1C_1)\, dx_1/dt_d &= -x_1 + x_2 \\
(R_1C_2)\, dx_2/dt_d &= x_1 - [1 + (R_1/R_2)]\, x_2 + (R_1/R_2)\, x_3 \qquad (2.1)\\
(R_1C_3)\, dx_3/dt_d &= (R_1/R_2)\, x_2 - [(R_1/R_2) + (R_1/R_3)]\, x_3
\end{aligned}$$

where the capacitor voltages were chosen as states and $t_d$ is dimensional time. Suppose that $C_2$ and $C_3$ are "parasitic," say $C_2=C_3=\varepsilon C_1$ and that all the resistors are of the same order of magnitude. Recognizing $R_1C_1$ as a typical large time constant and defining the slow dimensionless time $t=t_d/(R_1C_1)$ [32,33], (2.1) becomes

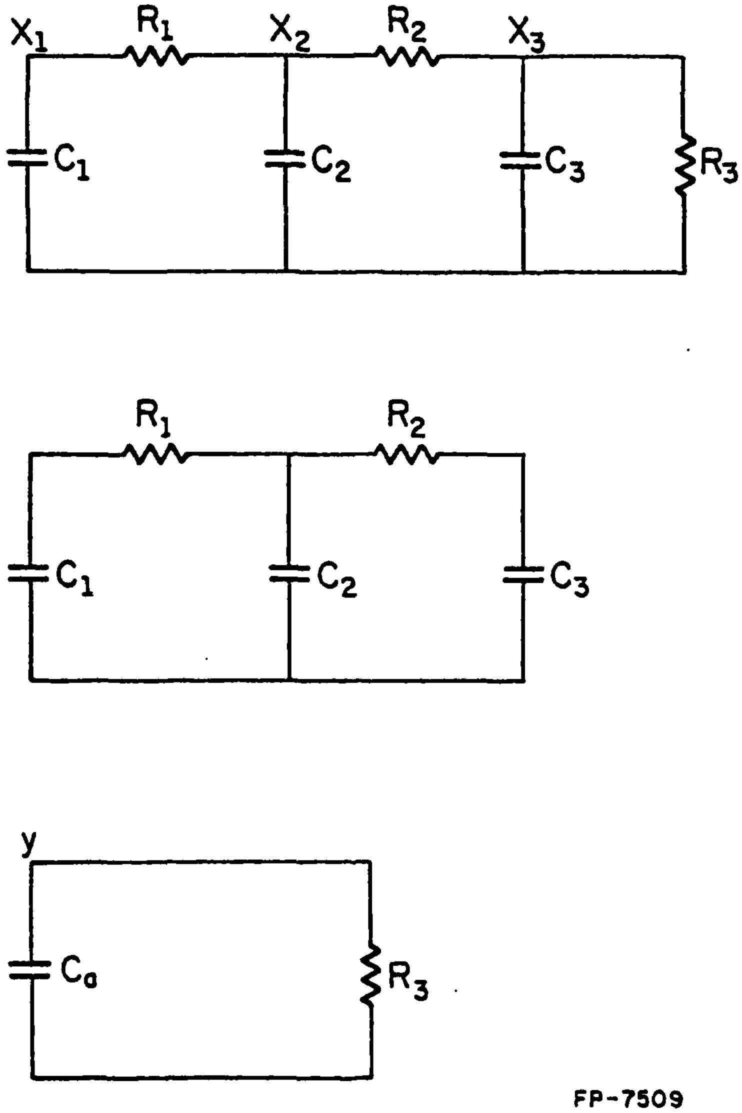


Fig. 2.1 (a) Circuit with $R_3$ much larger than $R_1$, $R_2$.
(b) Fast circuit described by (2.6).
(c) Slow reduced circuit for y in (2.33).

$$dx_1/dt = - x_1 + x_2$$
$$\varepsilon dx_2/dt = x_1 - [1 + (R_1/R_2)]\, x_2 + (R_1/R_2)\, x_3 \tag{2.2}$$
$$\varepsilon dx_3/dt = (R_1/R_2)\, x_2 - [(R_1/R_2) + (R_1/R_3)]\, x_3$$

with $\varepsilon$ multiplying the derivatives of $x_2$ and $x_3$. Thus $x_1$ appears as the y-variable and $x_2, x_3$ as the z-variables of (1.1) and the model is explicit because the two-by-two matrix of $x_2, x_3$ is nonsingular. The slow reduced model (1.2) represents the circuit with parasitic capacitors $C_2$ and $C_3$ opened (Fig. 2.1b), whereas in the fast reduced model (1.3) the large capacitor $C_3$ is shortened [30] (Fig. 2.1c).

In the same circuit nonexplicit singular perturbations occur when all capacitors are of the same order of magnitude, say $C_1=C_2=C_3=C$, but the resistors are not. For example, if $R_1$ and $R_2$ are small and $R_3$ is large, say

$$R_1 = r,\ R_2 = r/2,\ R_3 = R,\ r/R = \varepsilon \tag{2.3}$$

typical large and small time constants are RC and rC, respectively, and in the dimensionless time variables

$$t = \frac{t_d}{RC},\ \tau = \frac{t_d}{rC},\ \frac{t}{\tau} = \varepsilon \tag{2.4}$$

the circuit is described by

$$\varepsilon\ (dx/dt) = dx/d\tau = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2-\varepsilon \end{bmatrix} x \triangleq A(\varepsilon)\, x. \tag{2.5}$$

Note that $\varepsilon$ multiplies all the derivatives in the slow time-scale t and thus there are no explicit slow y-variables in the system. If $\frac{\partial g}{\partial z}$, that is A(0), were nonsingular, no slow phenomenon would exist in (2.5) and the system would not possess the two-time-scale property. However, A(0) is singular indicating the existence of a "hidden" slow phenomenon. To see this assume that $x(0) = [1 \;\; 1 \;\; 1]^T$ in (2.5). Then the slow-time derivatives dx/dt remain finite when $\varepsilon \rightarrow 0$ suggesting that (2.5) is a two-time-scale system. Physically the slow phenomenon is the discharge of the capacitors through the large "leakage" resistor $R_3$. Neglecting this "leakage," Fig. 2.1b, makes the slow phenomenon infinitely slow, that is, constant and corresponds to setting $\varepsilon$=0 in the $\tau$-model of (2.5)

$$\frac{dx}{d\tau} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix} x = A(0)x. \qquad (2.6)$$

Since A(0) is singular the equation

$$A(0)\, x = 0 \qquad (2.7)$$

has an infinite number of solutions given by

$$x = \alpha\, [1 \;\; 1 \;\; 1]^T \qquad (2.8)$$

where $\alpha$ is any real number, that is, (2.6) has a continuum of equilibrium points. This can be seen from the circuit of Fig. 2.1b where any x such that

$$x_1 - x_2 = 0,\ x_3 - x_2 = 0 \tag{2.9}$$

is an equilibrium point. The line represented by (2.8)-(2.9) will be denoted by S.

Kirchhoff's current law (KCL) applied to the ground node of Fig. 2.1b gives the dual property that is the conservation of total charge for all $\tau$,

$$C_1x_1(\tau) + C_2x_2(\tau) + C_3x_3(\tau) = C_1x_1(0) + C_2x_2(0) + C_3x_3(0) \tag{2.10}$$

which means that every trajectory $x(\tau)$ of (2.6) is confined to a plane F passing through the initial point $x(0)$ orthogonal to the vector $[C_1 \quad C_2 \quad C_3]^T$. The quantity in (2.10), constant when $\varepsilon=0$, becomes slowly varying when $\varepsilon > 0$, that is when the "leakage" $R_3$ is introduced. A circuit describing this slow phenomenon is given in Fig. 2.1c and will be derived in the next section.

From the above discussion we conclude that the trajectories $x(t)$ of the original system 2.5 consist of two distinct parts. First in a "boundary layer" near plane F the state $x(t)$ rapidly approaches line S. Then, from a neighborhood of the intersection of plane F with line S, $x(t)$ continues to slowly "slide" along line S. The geometry of this situation is sketched in Fig. 2.2. Note that the behavior of $x(t)$ is similar to that of the explicit model, that is, a fast transient is followed by a slow motion close to a line of "quasi-equilibria" S. The basic difference is that in the explicit model (2.2) the plane F close to which the boundary layer occurs, is orthogonal to axis $x_1$ (Fig. 2.3).

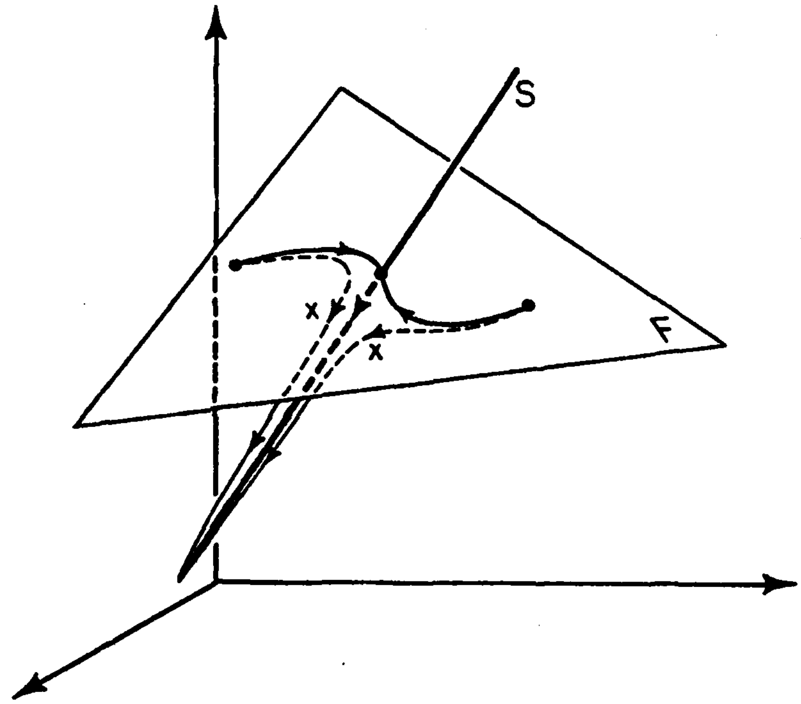


Fig. 2.2 Trajectories of the auxiliary circuit in Fig. 2.1b lie on F. Trajectories of the actual circuit Fig. 2.1a, are denoted by x.

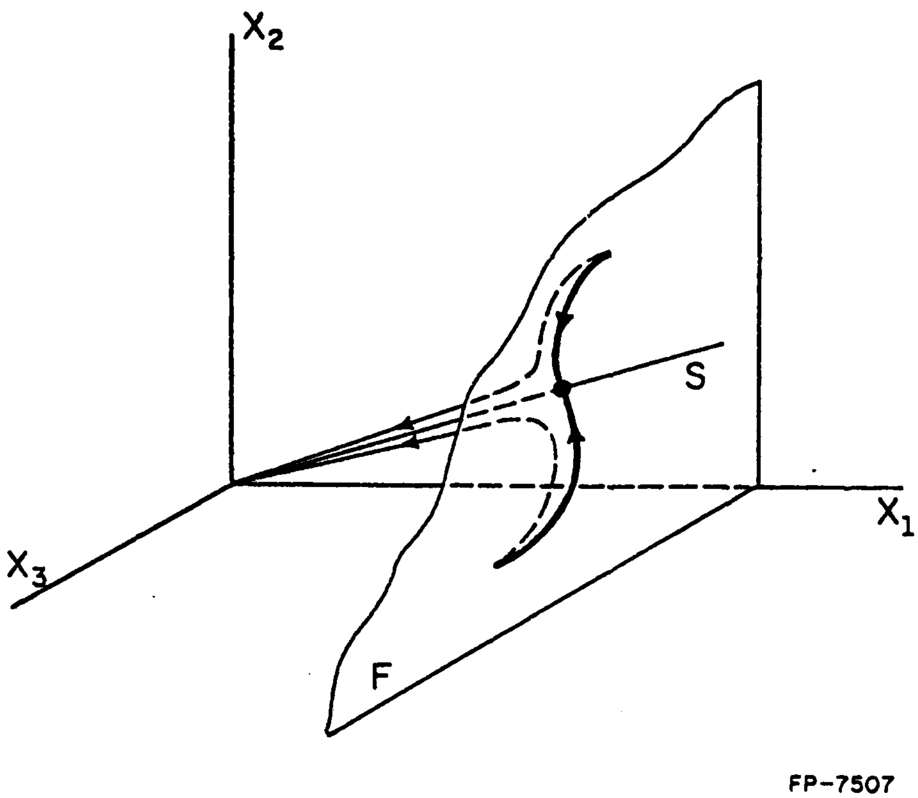


Fig. 2.3 Equilibrium (S) and dynamic manifolds (F) of the explicit model 2.2.

Examination of Fig. (2.2)-(2.3) indicates that in nonexplicit models fast dynamics are observed by all states whereas in explicit models they are only weakly observed by some states (the y-variables).

This example indicates that the time scales of the original system (2.5) are related to the equilibrium and conservation properties of the auxiliary system (2.6) in $\tau$-scale. These properties are coordinate free and characterize all two-time-scale systems reducible to the explicit model. In the next section they will serve for a choice of coordinates in which the time scales are explicit.

## 2.3 Nonexplicit Singularly Perturbed Systems

The discussion of the previous section will now be generalized to the system

$$\varepsilon dx/dt = dx/d\tau = A(\varepsilon)\, x \tag{2.11}$$

where $x \in R^n$, $A(\varepsilon)$ is a time invariant nxn matrix depending on $\varepsilon$, and t, $\tau$ the slow and fast time variables, respectively. The following is assumed about $A(\varepsilon)$.

Assumption 2.1 $A(\varepsilon)$ can be written as

$$A(\varepsilon) = A_o + \varepsilon\, A_1(\varepsilon) \tag{2.12}$$

with $A_1(\varepsilon)$ bounded at $\varepsilon$=0 and $A(0)=A_o$, satisfying

$$\mathcal{R}(A_o) \oplus \mathcal{N}(A_o) = R^n \quad (*) \tag{2.13}$$

(*) In the terminology of [13], (2.13)-(2.14) is equivalent to ind $A_o$=1. In [13] it is shown that this condition is necessary for $\lim_{\varepsilon \to 0} x(t)$ to exist.

where $\mathcal{R}(A_o)$ is the range space of $A_o$, $\mathcal{N}(A_o)$ is the null space of $A_o$, and $\oplus$ denotes the direct sum of two spaces [34]. The dimensions of $\mathcal{R}(A_o)$, $\mathcal{N}(A_o)$ are

$$\dim \mathcal{R}(A_o) = \rho \geq 1, \ \dim \mathcal{N}(A_o) = \nu \geq 1, \ \rho + \nu = n. \tag{2.14}$$

Equation (2.13) is equivalent to saying that $A_o$ has a complete set of eigenvectors corresponding to its zero eigenvalues, which in turn is equivalent to the following: $\mathcal{R}(A_o)$ is the invariant space (eigenspace) of $A_o$ corresponding to the nonzero eigenvalues, and $\mathcal{N}(A_o)$ is the invariant space (eigenspace) corresponding to the zero eigenvalues.

To study the time-scale behavior of (2.11) the auxiliary system

$$dx/d\tau = A_o\, x \tag{2.15}$$

is defined with $A_o$ as in (2.12). By assumption, (2.15) has a $\nu$-dimensional <u>equilibrium manifold</u>(*) (i.e. $\mathcal{N}(A_o)$) S consisting of all x such that

$$A_o\, x = 0. \tag{2.16}$$

If W is a $\rho$xn matrix, rank W=$\rho$, whose rows span the row space of A, then [34]

$$Wx = 0 \qquad\qquad \forall\ x \in S \tag{2.17}$$

To see the conservation property of (2.15) we note that if V is a $\nu$xn matrix, rank V = $\nu$ whose rows span the left null space of $A_o$, i.e. $VA_o = 0$, then

(*)Although S is presently, simply a subspace, we call it manifold in anticipation of the nonlinear extension in Chapter 3.

$$V(dx/d\tau) = V A_o x = 0 \qquad \forall \tau, \forall x(0) \in R^n. \tag{2.18}$$

Thus, the $\nu$-dimensional quantity Vx is constant along the trajectories of (2.15),

$$V x(\tau) = V x(0), \qquad \forall x(0) \in R^n. \tag{2.19}$$

This means that for each value of V x(0) the trajectory of (2.15) is confined to a linear manifold(*) defined by (2.19).

This linear manifold called dynamic manifold F, is orthogonal to the rows of V and contains the initial point x(0). The orthogonality between the left null space and the range space of a matrix [34] implies that F is a translate of $\mathcal{R}(A_o)$.

The above discussion has established equilibrium (Eq. (2.17)) and conservation (Eq. (2.19)) properties analogous to the ones of the RC-circuit of the previous section (Eq. (2.9) and (2.10)). The behavior of the trajectories is still the one depicted in Fig. 2.2 with S and F defined by (2.17) and (2.19), respectively. We are now ready to define a new set of coordinates in which the time scales are explicit.

Theorem 2.2 Under Assumption (2.1) the change of coordinates

$$y = Vx, \; z = Wx \tag{2.20}$$

transforms (2.11) into the explicit model (1.1) with $\bar{z}(t) = 0$.

(*)A linear manifold of dimension r is a translation of an r-dimensional subspace.

<u>Proof</u>: Since the rows of V,W form bases for the left null and row spaces of $A_o$, respectively, the transformation

$$T = \begin{bmatrix} V \\ W \end{bmatrix} \tag{2.21}$$

defined by (2.20) has inverse

$$T^{-1} = [P \quad Q] \tag{2.22}$$

where the columns of P,Q form bases of $\mathcal{N}(A_o)$, $\mathcal{R}(A_o)$. Hence,

$$T\left(\frac{A_o}{\varepsilon} + A_1(\varepsilon)\right)T^{-1} = \frac{1}{\varepsilon}\begin{bmatrix} V A_o P & V A_o Q \\ W A_o P & W A_o Q \end{bmatrix} + \begin{bmatrix} V A_1(\varepsilon)P & V A_1(\varepsilon)Q \\ W A_1(\varepsilon)P & W A_1(\varepsilon)Q \end{bmatrix}$$

$$= \begin{bmatrix} V A_1(\varepsilon)P & V A_1(\varepsilon)Q \\ W A_1(\varepsilon)P & \frac{W A_o Q}{\varepsilon} + W A_1(\varepsilon)Q \end{bmatrix} \tag{2.23}$$

$$\begin{aligned} dy/dt &= A_{11}(\varepsilon)\ y + A_{12}(\varepsilon)\ z \\ \varepsilon dz/dt &= \varepsilon\ A_{21}(\varepsilon)\ y + A_{22}(\varepsilon)\ z \end{aligned} \tag{2.24}$$

where $A_{11}(\varepsilon) \triangleq V\ A_1(\varepsilon)P$, $A_{12}(\varepsilon) \triangleq V\ A_1(\varepsilon)Q$, $A_{21}(\varepsilon) \triangleq W\ A_1(\varepsilon)P$ and

$$A_{22}(\varepsilon) \triangleq W\ A_oQ + \varepsilon\ W\ A_1(\varepsilon)Q. \tag{2.25}$$

To show that (2.24) is explicit model we need to show that $A_{22}(0)$ is nonsingular. Notice that Assumption (2.1) implies that $\mathcal{R}(A_o)$ is the

eigenspace of the nonzero eigenvalues of $A_o$. Hence there is a $\rho x \rho$ nonsingular matrix G whose eigenvalues are the nonzero eigenvalues of $A_o$ such that

$$A_o Q = QG. \tag{2.26}$$

The last relation implies

$$A_{22}(0) = W A_o Q = W Q G = G \tag{2.27}$$

which is nonsingular.

<u>Remark</u>: Writing

$$x = T^{-1} \begin{bmatrix} y \\ z \end{bmatrix} = Py + Qz \tag{2.28}$$

we see that y,z are the representations of x with respect to bases P,Q of $\mathcal{N}(A_o)$, $\mathcal{R}(A_o)$, respectively. Another way to view (2.28) is that Py is the projection of x on $\mathcal{N}(A_o)$ along $\mathcal{R}(A_o)$ and hence y is the representation of this projection with respect to basis P. A similar interpretation holds for Qz and z.

We now illustrate the application of Theorem 2.2 with the RC-circuit of Fig. 2.1 in which the time scales are due to large and small resistors as in (2.3). The auxiliary system is given in (2.6) and the equilibrium and dynamic manifolds are defined by (2.9) and (2.10), respectively. A choice of coordinates according to these equations is

$$y = (C_1 x_1 + C_2 x_2 + C_3 x_3)/C_a \tag{2.29}$$

$$z_1 = x_1 - x_2, \qquad z_2 = x_3 - x_2 \tag{2.30}$$

where the division by

$$C_a = C_1 + C_2 + C_3 \tag{2.31}$$

in (2.29) retains the physical meaning of y as a voltage variable. In the new coordinates the circuit is described by

$$\begin{aligned} dy/dt &= -(C/C_a)\,y + (C^2/C_a^2)\,z_1 - (2C^2/C_a^2)\,z_2 \\ \varepsilon(dz_1/dt) &= -2z_1 - 2z_2 \\ \varepsilon(dz_2/dt) &= -\varepsilon y - (1 - \varepsilon(C/C_a))\,z_1 - (4 + \varepsilon(2C/C_a))\,z_2. \end{aligned} \tag{2.32}$$

As stated in Theorem 2.2 the z-equations in (2.32) give $\bar{z}(t) = 0$. Hence the slow reduced model is

$$d\bar{y}/dt = -(C/C_a)\bar{y} \tag{2.33}$$

represented by the circuit in Fig. 2.1c and the fast reduced model is

$$\begin{aligned} d\tilde{z}_1/d\tau &= -2\tilde{z}_1 - 2\tilde{z}_2 \\ d\tilde{z}_2/d\tau &= -\tilde{z}_1 - 4\tilde{z}_2 \end{aligned} \tag{2.34}$$

represented by the circuit in Fig. 2.1b where the voltages with respect to the "reference" node 2 are used as states.

It is interesting to note the physical interpretation of the new variables. The slow variable y is proportional to the sum of the charges on the three capacitors and can be considered the voltage on the

"aggregate" capacitor $C_a$ (Eq. (2.31), Fig. 2.1c). Application of Kirchhoff's current law to the ground node of Fig. 2.1a shows that the time derivative of y is proportional to the current in $R_3$. Since $R_3$ is large dy/dt is small and y qualifies as a slow variable. The fast variable $z_1$ equals the voltage across $R_1$ which due to the smallness of $R_1$ diminishes quickly to values close to zero. Hence $z_1$ qualifies as a fast variable. A similar interpretation holds for $z_2$. In Chapter 4 we show that this selection of slow and fast variables is good for a wide class of nonlinear dynamic networks.

## 2.4 Nested Reduced Order Models

In the previous section we showed how equilibrium and conservation properties are used to transform a nonexplicit singularly perturbed model to an explicit one. Writing the explicit model (2.24) in the fast time $\tau$ and letting $\varepsilon \to 0$ we obtain the fast reduced model

$$\frac{d\tilde{z}}{d\tau} = A_{22}(0)\tilde{z} \qquad \tilde{z}(0) = z(0). \tag{2.35}$$

Similarly writing the model in the slow time t and letting $\varepsilon \to 0$ we obtain the slow reduced model

$$\frac{d\bar{y}}{dt} = A_{11}(0)\bar{y} \qquad \bar{y}(0) = y(0). \tag{2.36}$$

If the eigenvalues of $A_{22}(0)$ have negative real parts the responses of (2.35)-(2.36) are $O(\varepsilon)$ approximations of the response of (2.24) over bounded intervals [8-10]. If, in addition, the eigenvalues of the slow system matrix $A_{11}(0)$ have negative real parts the approximation is valid over unbounded intervals [36].

It may happen, however, that some eigenvalues of $A_{11}(0)$ are zero. In these cases the approximation is not valid over unbounded intervals since the response of (2.36) tends to a nonzero constant whereas the response of (2.24) tends to zero. Treatment of (2.24) as a two-time-scale system is inadequate because the system may have more than two time scales. Simple expansions of the eigenvalues and eigenvectors, for example, show that if $A_{11}$ is singular the matrix

$$\begin{bmatrix} \varepsilon A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{bmatrix}$$

which is the matrix of (2.24) in the fast time scale with $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$, independent of $\varepsilon$, has $O(\varepsilon^2)$ eigenvalues in addition to $O(1)$ and $O(\varepsilon)$ ones.

Instead of treating the original system (2.11) as a multi-time-scale one, we prefer to deal with only two time scales at a time. That is, starting from the fastest time $t_1 \triangleq \tau$ we consider the system operating in scale $t_1$ and $t_2 = \varepsilon t_1$ only. Viewed from $t_1$ speeds are $O(1)$ and the rest are $o(1)$; we do not specify whether they are $O(\varepsilon)$, $O(\varepsilon^2)$, etc. Changing time scales to the slower $t_2$, some speeds are $o(\frac{1}{\varepsilon})$ and, in time-scale $t_2$, are assumed to reach their quasi-steady-state instantaneously. The rest of the system is again treated as two-time-scale with $t_2$ the fast time.

To make this idea precise we employ the block diagonalizing transformation [37,38]

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} I - \varepsilon^2 HL & -\varepsilon H \\ \varepsilon L & I \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \tag{2.37}$$

where L,H satisfy

$$A_{21}(\varepsilon) + \varepsilon\, L\, A_{11}(\varepsilon) - A_{22}(\varepsilon)\, L - \varepsilon^2\, L\, A_{12}(\varepsilon)\, L = 0 \tag{2.38}$$

and

$$A_{12}(\varepsilon) - H\, (A_{22}(\varepsilon) + \varepsilon^2\, L\, A_{12}(\varepsilon)) + \varepsilon(A_{11}(\varepsilon) - \varepsilon\, A_{12}(\varepsilon)\, L) = 0. \tag{2.39}$$

In the $\xi,\eta$ coordinated (2.24) becomes

$$\frac{d\xi}{dt} = (A_{11}(\varepsilon) - \varepsilon\, A_{12}(\varepsilon)\, L)\xi \tag{2.40}$$

$$\varepsilon\, \frac{d\eta}{dt} = (A_{22}(\varepsilon) + \varepsilon^2\, L\, A_{12}(\varepsilon))\eta\ . \tag{2.41}$$

Since $A_{22}(0)$ is nonsingular (2.35) is a regular perturbation of (2.41) and can be used as a fast reduced order model. However, if $A_{11}(0)$ is singular satisfying Assumption 2.1, (2.40) is nonexplicit singularly perturbed model in the form (2.11). Hence, arguing as in Section 2.3, we apply transformation (2.20) to define slow and fast variables in time scale $t_2$. The process can be repeated until all time scales are "peeled off" defining nested reduced order models.

## 2.5 Systems With Inputs

Although some control design work has been done for generalized singularly perturbed systems [13,14], most of the literature [2-4,39-40] deals with the explicit model

$$\begin{aligned} \dot{y} &= A_{11}\, y + A_{12}\, z + B_1\, u \\ \varepsilon\dot{z} &= A_{21}\, y + A_{22}\, z + B_2\, u \end{aligned} \tag{2.42}$$

where $A_{22}$ is nonsingular. Note that in the y-equations the gain of the control u is $O(1)$ whereas in the z-equations it is $O(\frac{1}{\varepsilon})$. We are interested in conditions under which the nonexplicit model

$$\varepsilon\frac{dx}{dt} = \frac{dx}{d\tau} = [A_o + \varepsilon A_1(\varepsilon)]x + [B_o + \varepsilon\overline{B}(\varepsilon)]\ u \tag{2.43}$$

where $A_o$ satisfies Assumption 2.1 and $\overline{B}(\varepsilon)$ is differentiable at $\varepsilon=0$, can be transformed to the explicit model (2.42).

Assumption 2.3 Let V be a $\nu$xn matrix that spans the left null space of $A_o$. Then

$$V\ B_o = 0 \tag{2.44}$$

that is, V is in the left null space of $B_o$.

Corollary 2.4 Under Assumptions 2.1, 2.3 the transformation

$$y = Vx, \qquad z = Wx \tag{2.45}$$

transforms (2.43) into the explicit model (2.42) with $\overline{z} = -A_{22}^{-1}(0)B_2(0)\overline{u}$ where $B_2(\varepsilon) = WB_o + \varepsilon W\overline{B}(\varepsilon)$ and $\overline{u}$ is the slow control.

Proof: Follows directly from Theorem 2.2 and (2.44).

Condition (2.44) essentially requires that the control driving the slow variable y have O(1) gain, as in the explicit model (2.42). If this condition is not met the variable y, slow in the free system (2.11), is subjected to high gain control altering the time-scale behavior of the system. We will have more to say about high gain feedback in the next chapter. Condition (2.44) is likely to be satisfied in well defined physical problems as demonstrated by the following example.

Consider the transformer of Fig. 2.4 where $L_1$, $L_2$ are the self-inductances of the coils, M is the mutual inductance and $x_1$, $x_2$ are the currents through the coils. Using $x_1$, $x_2$ as states, the state description of the system is

$$\begin{aligned} dx_1/dt &= -(R_1L_2/d)x_1 + (MR_2/d)x_2 - (L_2/d)\, v_i \\ dx_2/dt &= (MR_1/d)x_1 - (R_2L_1/d)x_2 + (M/d)\, v_i \end{aligned} \qquad (2.46)$$

where $d = L_1L_2 - M^2$.

In the case of an ideal transformer, $d = L_1L_2 - M^2 = 0$. For nonideal transformer with small leakage

$$\frac{L_1L_2 - M}{L_1L_2} = \varepsilon \qquad (2.47)$$

where $\varepsilon$ is a small positive parameter. Using (2.47) the system matrix of (2.46) becomes

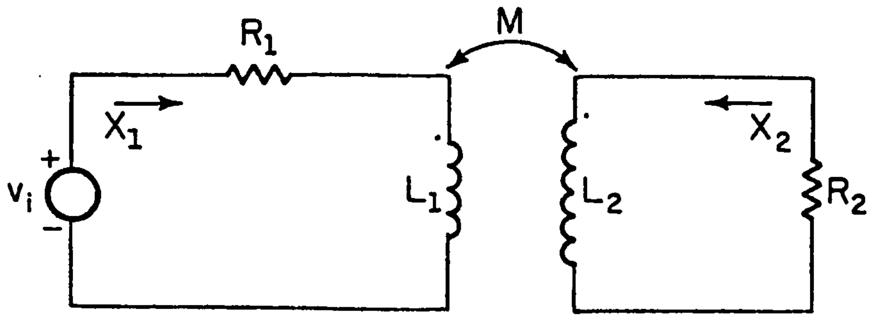


Fig. 2.4 A nonideal transformer with small leakage.

$$A(\varepsilon) = \frac{1}{\varepsilon}\begin{bmatrix} -\frac{R_1}{L_1} & \frac{\sqrt{1-\varepsilon}\; R_2}{\sqrt{L_1L_2}} \\ \frac{\sqrt{1-\varepsilon}\; R_1}{\sqrt{L_1L_2}} & -\frac{R_2}{L_2} \end{bmatrix} \tag{2.48}$$

or, substituting $\sqrt{1-\varepsilon} = 1 - \frac{\varepsilon}{2} + O(\varepsilon^2)$

$$A(\varepsilon) = \frac{1}{\varepsilon}\,(A_o + \varepsilon A_1(\varepsilon)) \tag{2.49}$$

where

$$A_o = \begin{bmatrix} -\frac{R_1}{L_1} & \frac{R_2}{\sqrt{L_1L_2}} \\ \frac{R_1}{\sqrt{L_1L_2}} & -\frac{R_2}{L_2} \end{bmatrix} . \tag{2.50}$$

The left null space of $A_o$

$$V = [L_1 \quad \sqrt{L_1L_2}] \tag{2.51}$$

and the row space of $A_o$

$$W = [R_1 \quad -R_2\sqrt{\frac{L_1}{L_2}}\,] \tag{2.52}$$

define, according to (2.45), the slow variable

$$y = L_1x_1 + \sqrt{L_1L_2}\;\; x_2 \tag{2.53}$$

and the fast variable

$$z = R_1x_1 - R_2\sqrt{\frac{L_1}{L_2}}\;\; x_2 \;. \tag{2.54}$$

In the new coordinates y,z the state equations become

$$\frac{dy}{dt} = - [1/(T_1+T_2)]y + [(T_1-T_2)/2(T_1+T_2)]z - \frac{v_i}{2}$$

$$\varepsilon\frac{dz}{dt} = \varepsilon[(\frac{1}{T_2} - \frac{1}{T_1})/2(T_1+T_2)]y - [\frac{1}{T_1} + \frac{1}{T_2} - \frac{\varepsilon}{T_1+T_2}]z \qquad (2.55)$$

$$- (\frac{1}{T_1} + \frac{1}{T_2}) v_i$$

where $T_1 = L_1/R_1$, $T_2 = L_2/R_2$ are the time constants of the primary and secondary R-L circuits. The slow model is obtained by setting $\varepsilon=0$ in the second equation of (2.55) and substituting the quasi-steady-state

$$\bar{z} = - v_i \qquad (2.56)$$

into the first equation giving

$$\frac{d\bar{y}}{dt} = [- \frac{1}{T_1+T_2}]\bar{y} - [\frac{T_1}{T_1+T_2}] v_i \ . \qquad (2.57)$$

The fast model is obtained by writing the second equation of (2.55) in the fast time-scale and setting $\varepsilon=0$

$$\frac{d\tilde{z}}{d\tau} = - (\frac{1}{T_1} + \frac{1}{T_2})\tilde{z} - (\frac{1}{T_1} + \frac{1}{T_2}) v_i \ . \qquad (2.58)$$

It is interesting to note the physical interpretation of the new variables y,z and of (2.54). By writing

$$y = L_1x_1 + \sqrt{L_1L_2}\, x_2 = L_1x_1+Mx_2 + O(\varepsilon) = \Phi_{11} + \Phi_{12} + O(\varepsilon) \qquad (2.59)$$

we see that y is, to $O(\varepsilon)$, the total flux linkage $\Phi_{11} + \Phi_{12}$ in coil 1. Aggregate physical quantities such as total flux linkage, total charge, total momentum etc., are often slow variables.

Noticing that $R_2x_2$ is the voltage $v_2$ across the secondary winding of the transformer and using (2.54, (2.56) we obtain as $\varepsilon \to 0$

$$v_2 = -(v_i+R_1x_1)\sqrt{\frac{L_2}{L_1}} = -(v_i+R_1x_1)\frac{N_2}{N_1} \tag{2.60}$$

where $N_1, N_2$ are the number of turns in coils 1 and 2. When the voltage drop $R_1x_1$ is small compared to $v_i$, which is usually the case, (2.60) reduces to the corresponding relation $v_2 = -\frac{N_2}{N_1} v_i$ for a leakage-free transformer.

Writing $M = \sqrt{L_1L_2}\ \sqrt{1-\varepsilon} \simeq \sqrt{L_1L_2}\ (1 - \frac{\varepsilon}{2})$ we see that, for this example, Assumption 2.3 is satisfied.

## 2.6 Structured Singularly Perturbed Forms

Section 3 dealt with the nonexplicit singularly perturbed form (2.11), in which all the states are, generally, mixed. However, it is well known that a more structured system matrix implies that state $x_1$ is predominantly slow. In this section, the methodology developed in Section 3 is used to study two other structured forms, the fast separated form and the weak connection form.

A system is said to be in the <u>fast separated singularly perturbed form</u> if

$$\varepsilon \begin{bmatrix} \frac{dx_1}{dt} \\ \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \frac{dx_1}{d\tau} \\ \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} \varepsilon A_{11} & A_{12} \\ \\ \varepsilon A_{21} & A_{22} \end{bmatrix} \tag{2.61}$$

where $x_1 \varepsilon R^{n1}$, $x_2 \varepsilon R^{n2}$, $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$ are matrices of appropriate dimensions and $A_{22}$ is a nonsingular matrix. Writing

$$\begin{bmatrix} \varepsilon A_{11} & A_{12} \\ \\ \varepsilon A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & A_{12} \\ \\ 0 & A_{22} \end{bmatrix} + \varepsilon \begin{bmatrix} A_{11} & 0 \\ \\ A_{21} & 0 \end{bmatrix} \tag{2.62}$$

we obtain

$$A_o = \begin{bmatrix} 0 & A_{12} \\ \\ 0 & A_{22} \end{bmatrix} \tag{2.63}$$

which satisfies Assumption 1. Since $A_{22}$ is nonsingular the range and row spaces have dimensions $n_2$ whereas the left and right null spaces have dimension $n_1$. It can be verified that

$$V = [I_{n_1} \quad -A_{12}A_{22}^{-1}] \tag{2.64}$$

$$W = [0 \quad I_{n_2}] \tag{2.65}$$

span the left null and row spaces of $\acute{A}_o$, respectively, and that

$$P = \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} A_{12} & A_{22}^{-1} \\ & I_{n_2} \end{bmatrix} \tag{2.66}$$

span the null and range spaces of $A_o$, respectively. Morover they satisfy (2.13).

<u>Corollary 2.5</u> If $A_{22}$ is nonsingular, then (i) the change of coordinates

$$y = x_1 - A_{12}A_{22}^{-1}\, x_2, \qquad z = x_2 \tag{2.67}$$

transforms (2.61) into the explicit model (1.1) with $\bar{z}(t) = 0$. (ii) The slow reduced model of (2.61) is

$$\frac{d\bar{y}}{dt} = (A_{11} - A_{12}A_{22}^{-1}\, A_{21})\bar{y} \tag{2.68}$$

and the fast reduced model is

$$\frac{d\tilde{z}}{d\tau} = A_{22}\, \tilde{z} \tag{2.69}$$

(iii) The state $x_2$ of (2.61) is predominantly fast whereas $x_1$ is mixed.

<u>Proof</u>: (i) follows directly from Theorem 2.2, using (2.64), (2.65). (ii) is obtained by bearing in mind that $\bar{z}(t) = 0$. (iii) follows by inverting (2.67)

$$x_1 = y + A_{12}A_{22}^{-1}\, z, \quad , \qquad x_2 = z \quad . \tag{2.70}$$

Note that state $x_2$ (=z) can be used in the fast reduced model (2.69) justifying the name "fast separated."

We now turn to the weak connection form which arises naturally in dynamic networks made of weakly connected "areas" [24-29]. A system is said to be in the <u>weak connection form</u> if[(*)]

$$\varepsilon \begin{bmatrix} \dfrac{dx_1}{dt} \\ \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \dfrac{dx_1}{d\tau} \\ \\ \dfrac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} A_{11}+\varepsilon\hat{A}_{11} & \varepsilon A_{12} \\ \\ \varepsilon A_{21} & A_{22}+\varepsilon\hat{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} \tag{2.71}$$

where $x_1$ and $x_2$ are $n_1$- and $n_2$- vectors, $A_{11}$, $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$, $A_{22}$ are matrices of appropriate dimensions and $A_{11}$, $A_{22}$ are singular matrices with a complete set of eigenvectors corresponding to the zero eigenvalues, that is, satisfying

$$\mathcal{R}(A_{11}) \oplus \mathcal{N}(A_{11}) = R^{n_1}, \qquad \mathcal{R}(A_{22}) \oplus \mathcal{N}(A_{22}) = R^{n_2} \tag{2.72}$$

with

$$\dim \mathcal{R}(A_{11}) = \rho_1 \geq 1,\ \dim \mathcal{N}(A_{11}) = \nu_1 \geq 1,\ \rho_1 + \nu_1 = n_1 \tag{2.73}$$

$$\dim \mathcal{R}(A_{22}) = \rho_2 \geq 1,\ \dim \mathcal{N}(A_{22}) = \nu_2 \geq 1,\ \rho_2 + \nu_2 = n_2\ . \tag{2.74}$$

Writing

$$\begin{bmatrix} A_{11}+\varepsilon\hat{A}_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22}+\varepsilon\hat{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} + \varepsilon \begin{bmatrix} \hat{A}_{11} & A_{12} \\ A_{21} & \hat{A}_{22} \end{bmatrix} \tag{2.75}$$

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(*) For convenience we deal with only two weakly connected "areas" but the ideas are directly applicable to any number of areas.

we obtain

$$A_o = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \tag{2.76}$$

which, because of (2.72), satisfies Assumption 2.1. Let $V_i$, $W_i$ span the left null and row spaces of $A_{ii}$ and $P_i$, $Q_i$ span the null and range spaces, respectively, i=1,2, satisfying

$$V_iP_i = I\nu_i, \quad W_iQ_i = I\rho_i, \qquad i=1,2 \ . \tag{2.77}$$

Corollary 2.6 Under assumption (2.72), (i) the change of coordinates

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} V_1x_1 \\ V_2x_2 \end{bmatrix} \quad , \quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_1x_1 \\ W_2x_2 \end{bmatrix} \tag{2.78}$$

transforms (2.71) into the explicit model (1.1) with $\overline{z}(t) = 0$. (ii) The slow reduced model is

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} V_1\ \hat{A}_{11}\ P_1 & V_1\ A_{12}\ P_2 \\ V_2\ A_{21}\ P_1 & V_2\ \hat{A}_{22}\ P_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{2.79}$$

and the fast reduced model is

$$\frac{d}{d\tau}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_1A_{11}Q_1+\varepsilon W_1\hat{A}_{11}Q_1 & \varepsilon W_1A_{12}Q_2 \\ \varepsilon W_2A_{21}Q_1 & W_2A_{22}Q_2+\varepsilon W_2\hat{A}_{22}Q_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} . \tag{2.80}$$

Proof: (i) Apply Theorem 2.2 noting that

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} , \quad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \tag{2.81}$$

span the left null and row spaces of $A_o$, respectively. (ii) is obtained by keeping in mind that $\bar{z}(t) = 0$.

There are few interesting points to be noted. From (2.78) a slow vector $y_1$ and a fast vector $z_1$ are defined in terms of area $x_1$ only; similarly for $x_2$. From (2.80) the fast variables $z_1, z_2$ are only weakly connected to each other and since $W_i A_{ii} Q_i$, $i=1,2$ are nonsingular these connections can be neglected for an $O(\varepsilon)$ approximation. The fundamental difference between the original (2.71) and the transformed system (2.80) is that (2.80) no longer has a continuum of equilibrium points. Hence, each area defines a local fast model $z_i$ connected with $O(\varepsilon)$ connections to other local models, whereas contributions $y_i$ from each area form a "slow core" describing the system-wide dynamics of (2.71). It is shown in Chapters 3 and 4 that this decomposition carries over to nonlinear weakly connected systems.

# CHAPTER 3

# NONEXPLICIT SINGULAR PERTURBATIONS IN NONLINEAR SYSTEMS

## 3.1 Introduction

An asymptotic procedure for time-scale separation in nonlinear models is of paramount importance since a nonlinear analog of the algebraic transformation in [37,38,41] is not available. In this chapter we give such a procedure and demonstrate its application to classes of nonlinear systems. The coordinate free characterization of singular perturbations [16] is extended to nonlinear systems of the form $\varepsilon\dot{x}=h(x,\varepsilon)$ and it is shown that equilibrium and conservation properties lead to a definition of new coordinates in which time scales are explicit (Section 3.2). In Section 3.3 we study a class of nonlinear high gain feedback control systems in which the controls enter linearly through a constant matrix but the open loop system, the output map and the feedback law may all be nonlinear. It is shown that these systems can be studied through singular perturbation techniques after they are transformed to the explicit form using the method of Section 3.2. The last section, 3.4, is devoted to interconnected systems whose isolated subsystems posses equilibrium and conservation properties. It is shown that in such systems weak connections give rise to two-time-scale behavior. Separation of the time scales defines a slow "core" which describes the system-wide behavior and a set of fast "residues" describing the local behavior of each subsystem. The decomposition, known for specific classes of linear models such as Markov chains [42-43] linearized models of power systems [24-28,44], electrical networks [44,45] and economic

systems [46] where it appeared for the first time, is established for a wide class of nonlinear systems with common features (i) equilibrium and conservation properties and (ii) weak connections between subsystems. Using this decomposition we give decentralized stability cirteria for this class of systems, analogous to those in [47-49].

## 3.2 Conservation and Equilibrium Properties in Nonlinear Systems

The need for coordinate free characterization of time scales in nonlinear systems is more pressing than in linear systems. Wide separation of eigenvalues provides some characterization in linear systems but the notion of modes is nonexistent in nonlinear systems. It will be shown in this section that the conservation and equilibrium properties introduced in Chapter 2 for linear systems can naturally be extended to nonlinear systems and that they lead to a new set of variables in which the time scales are explicit.

To motivate the discussion we re-examine the explicit model (1.1) from a different point of view. Writing (1.1) in the fast time scale

$$\begin{aligned} \frac{dy}{d\tau} &= \varepsilon f(y,z,\varepsilon) \\ \frac{dz}{d\tau} &= g(y,z,\varepsilon) \end{aligned} \tag{3.1}$$

and setting $\varepsilon=0$ we obtain the auxiliary system

$$\begin{aligned} \frac{dy}{d\tau} &= 0 \\ \frac{dz}{d\tau} &= g(y,z,0) \end{aligned} \tag{3.2}$$

which has the following two important properties.

<u>Conservation Property</u> A function of the state

$$\sigma(y,z) = y \tag{3.3}$$

remains at its initial value $\sigma(y(0), z(0)) = y(0)$, that is, it is conserved during the motion of (3.2).

<u>Equilibrium Property</u> System (3.2) possesses a set of nonisolated (continuum) equilibrium points defined by

$$g(y,z,0) = 0 \; . \tag{3.4}$$

The equilibria defined by (3.4) are the "quasi-steady-states" to which the fast transients of (3.1) converge as explained in Chapter 1.

A generalized version of (3.1) is a system in the form

$$\varepsilon \; \frac{dx}{dt} = \frac{dx}{d\tau} = h(x,\varepsilon) \tag{3.5}$$

which in $\tau$-scale at $\varepsilon$=0

$$\frac{dx}{d\tau} = h(x,0) \tag{3.6}$$

has equilibrium and conservation properties analogous to the properties of (3.1). System (3.5) is studied in a domain $D \subset R^n x[0,\varepsilon_o]$ in which function h is assumed to be continuously differentiable with respect to x and $\varepsilon$.

Assumption 3.1 System (3.5) satifies the following conditions for existence of manifolds(*) S and F.

Equilibrium Manifold S The set

$$S = \{x | h(x,0)=0, \ x \in D\} \tag{3.7}$$

defines a $\nu$-dimensional differentiable manifold, $\nu \geq 1$. Hence, there exists continuously differentiable function $\varphi: R^n \to R^\rho$, $\rho = n - \nu$, rank $\frac{\partial\varphi}{\partial x} = \rho$, $\forall \ x \in D$ such that

$$\varphi(x)=0 \iff h(x,0)=0 \tag{3.8}$$

that is, in the domain of interest D, every equilibrium of (3.6) satisfies $\varphi(x)=0$ and every x satisfying $\varphi(x)=0$ is an equilibrium of (3.6).

Dynamic Manifold $F_{x_o}$ There exists continuously differentiable function $\sigma: R^n \to R^\nu$ such that for each $x(0)=x_o$ the $\rho$-dimensional ($\rho=n-\nu$) manifold

$$F_{x_o} = \{x | \sigma(x) - \sigma(x_o)=0, \quad \text{rank} \ \frac{\partial\sigma}{\partial x} = \nu\} \tag{3.9}$$

in an invariant manifold of (3.6) that is a trajectory originating in $F_{x_o}$ remains in $F_{x_o}$

$$\sigma(x(\tau)) - \sigma(x_o)=0, \qquad \forall \ \tau \geq 0 \ . \tag{3.10}$$

(*) Manifolds are generalizations to $R^n$ of objects such as curves and surfaces in $R^3$. More precisely, let $\Phi: R^n \to R^m$ be a continuously differentiable function from $R^n$ into $R^m$. Then if the set $M = \{x | \Phi(x)=0$ and $\frac{\partial}{\partial x} \Phi(x)$ has rank m$\}$ is nonempty, it is an r-dimensional manifold, r=n-m [50,51].

Moreover, for all $x_o \in D$, manifolds S and $F_{x_o}$ are not tangent to each other, that is, for all x in the intersection of S and $F_{x_o}$

$$, \quad \text{rank} \begin{bmatrix} \frac{\partial\varphi}{\partial x} \\ \\ \frac{\partial\sigma}{\partial x} \end{bmatrix} = \eta \quad . \tag{3.11}$$

<u>Theorem 3.2</u> Under assumption 3.1, the change of coordinates

$$y=\sigma(x) \qquad , \qquad z=\varphi(x) \tag{3.12}$$

transforms (3.5) into the separated explicit model (1.1) with $\left.\frac{\partial g}{\partial z}\right|_{\varepsilon=0}$ nonsingular and $\bar{z}(t) = 0$.

<u>Proof</u>: Differentiating (3.10) and using (3.6) we have

$$\frac{\partial\sigma}{\partial x} h(x,0) = 0 \quad . \tag{3.13}$$

Differentiating $y=\sigma(x)$ and using the mean value theorem in $\varepsilon$ for each component of h

$$\frac{dy}{dt} = \frac{1}{\varepsilon}\frac{\partial\sigma}{\partial x} h(x,\varepsilon) = \frac{\partial\sigma}{\partial x}\frac{\partial h}{\partial\varepsilon} \tag{3.14}$$

we see that y is the slow variable of (1.1). Using the inverse transformation $x=\gamma(y,z)$ of (3.12) which exists because of (3.11), and differentiating $z=\varphi(x)$ we obtain

$$\varepsilon\frac{dz}{dt} = \frac{\partial\varphi}{\partial x} h(x,\varepsilon) = \frac{\partial\varphi}{\partial x} h(\gamma(y,z),\varepsilon) \triangleq g(y,z,\varepsilon) \quad . \tag{3.15}$$

We show that $(\partial g/\partial z)\Big|_{\varepsilon=0}$ is nonsingular by contradiction. Assuming that it is singular the equilibrium manifold of (3.14)-(3.15) has dimension greater than $\nu$ which is contradiction because (3.12) is a nonsingular transformation. Finally from $x \in S \iff x=\gamma(y,0)$ it follows that $h(\gamma(y,0),0)=0$ and

$$g(y,0,0)=0 \tag{3.16}$$

implying that $\bar{z}(t) = 0$.

The intuitive idea behind this theorem is illustrated by Fig. 3.1. If the equilibrium manifold S is attractive, the trajectories of (3.6) which are confined to some F due to (3.10), converge to S and when $\tau \to \infty$ they terminate at the intersection of F and S. Instead the trajectories of (3.5) rapidly approach S staying in a boundary layer close to F and then slowly continue their motion remaining close to S. Since the trajectories are initially close to F the quantity $\sigma(x)$ stays almost constant during this interval; thus it qualifies as a predominantly slow variable. On the other hand, the quantity $\varphi(x)$, which is large away from S where the trajectory starts, rapidly diminishes when the trajectory approaches S; thus it qualifies as a predominantly fast variable.

As an illustration we consider (3.17)

$$\begin{aligned}
\frac{dx_1}{d\tau} &= -\varphi_1(x) + (x_1+x_3)\,\varphi_2(x) - \varepsilon x_1 \\
\frac{dx_2}{d\tau} &= -2x_2\,\varphi_2(x) - \varepsilon x_2^3 \\
\frac{dx_3}{d\tau} &= \varphi_1(x) + (x_1+x_3)\,\varphi_2(x) - \varepsilon x_3
\end{aligned} \tag{3.17}$$

over $R^3 \supset D = \{(x_1,x_2,x_3) | \ x_1 > 1, \ x_2 > 0.5, \ x_3 > 0.5\}$ where $\varphi_1(x),\varphi_2(x)$ are continuously differentiable functions defined over D. Setting $\varepsilon=0$ in (3.17) we obtain

$$\frac{dx_1}{d\tau} = -\varphi_1(x) + (x_1+x_3)\varphi_2(x)$$

$$\frac{dx_2}{d\tau} = -2x_2\varphi_2(x) \tag{3.18}$$

$$\frac{dx_3}{d\tau} = \varphi_1(x) + (x_1+x_3)\varphi_2(x)$$

for which

$$\varphi_1(x)=0 \qquad , \qquad \varphi_2(x)=0 \tag{3.19}$$

define the equilibrium manifold S. It is easily verified that the dynamic manifolds are defined by $\sigma(x) = \sigma(x(0))$ where

$$\sigma(x) = (x_1+x_3)x_2 \ . \tag{3.20}$$

The equilibrium manifold and a dynamic manifold of this system are shown in Fig. 3.1 where functions $\varphi_1,\varphi_2$ were chosen as

$$\varphi_1(x) = x_1-x_3 \qquad , \qquad \varphi_2(x) = x_2^2-x_3-x_1+1 \ . \tag{3.21}$$

## 3.3 High Gain Feedback and Disturbance Rejection

Use of high gain in feedback loops has been known to reduce the effects of disturbances, parameter variations and distortions [19,52]. Early investigations using root locus techniques [53] have shown that

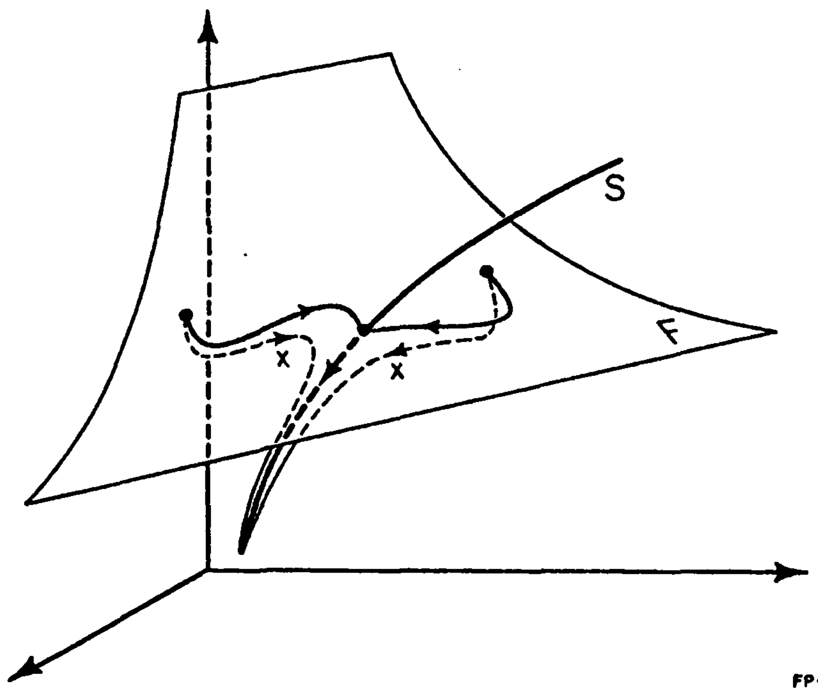


Fig. 3.1 Equilibrium (S) and dynamic (F) manifolds of (3.18).

under high gain some poles of the closed loop system tend to infinity, a characteristic of singularly perturbed systems. Similar behavior is exhibited by multivariable systems [54]. An extensive study of high gain in Linear Time Invariant (LTI) systems was undertaken in [17,55] where it was pointed out that every high gain system is a singularly perturbed one and vice versa. In this section we show that the relation between high gain feedback and singular perturbations extends beyond the class of LTI systems.

We consider the system

$$\begin{aligned} \dot{x} &= f(x)+Bu \\ y &= g(x) \end{aligned} \tag{3.22}$$

under the output feedback

$$u = \frac{1}{\varepsilon}k(y) \tag{3.23}$$

and we study the behavior of the closed loop system when the gain $\frac{1}{\varepsilon}\to\infty$. The state x is a n-vector and the input and output vectors both have dimension $m < n$. Functions f,g are defined in a domain $D_x \subset R^n$ and function k is defined in a domain $D_y \subset R^m$. All functions are assumed to be differentiable a sufficient number of times. Moreover we make the following basic assumption.

Assumption 3.3 (a) Matrix B is full rank. Hence, there exists a $\nu$xn ($\nu$=n-m) matrix V, rank V=$\nu$, such that VB=0.

(b) There exists a unique set point $y^* \in Dy$ such that

$$k(y^*) = 0 \tag{3.24}$$

(c) Matrix $\begin{bmatrix} V \\ g_x \end{bmatrix}$, where $g_x$ is the partial derivative of g, is nonsingular $\forall\, x \in D_x$ satisfying $g(x)-y^* = 0$.

Part b of the assumption could be relaxed to allow isolated roots of (3.24). However, nonisolated roots which imply dead zones are excluded. When the output is linear in x, y=Cx, Assumption 3.3(c) requires $\begin{bmatrix} V \\ C \end{bmatrix}$ to be nonsingular which is equivalent to the assumption of [17,55] that CB is nonsingular. Indeed if Assumption 3.3(c) holds $\begin{bmatrix} V \\ C \end{bmatrix} B = \begin{bmatrix} 0 \\ CB \end{bmatrix}$ must be full rank and $(CB)^{-1}$ exists; conversely if $(CB)^{-1}$ exists the row space of C and the left null space of B are disjoint and $\begin{bmatrix} V \\ C \end{bmatrix}$ is nonsingular.

<u>Theorem 3.4</u> Under Assumption 3.3 the transformation

$$\begin{aligned} y_s &= Vx \qquad (*) \\ z &= g(x)-y^* \end{aligned} \tag{3.25}$$

transforms the high gain system (3.22)-(3.23) into an explicit singularly perturbed form with $\bar{z}=0$.

<u>Proof</u>: Substituting (3.23) into (3.22) and rescaling the time, $\tau=t/\varepsilon$ we obtain

$$\frac{dx}{d\tau} = \varepsilon f(x) + Bk(g(x)) \tag{3.26}$$

whose auxiliary system is

$$\frac{dx}{d\tau} = Bk(g(x)) \tag{3.27}$$

---

*We temporarily change notation letting $v_s$ instead of y denote the slow variable. The latter has been reserved for its more traditional use as an output variable.

Since by Assumption 3.3(a)

$$V\frac{dx}{d\tau} = VBk(g(x)) = 0 \tag{3.28}$$

the relation

$$\sigma(x) = Vx, \qquad \sigma(x)=\sigma(x(0)) \tag{3.29}$$

defines the family of m-dimensional dynamic manifolds. By Assumption 3.3(b) all $x \in D_x$ satisfying

$$g(x)=y^* \tag{3.30}$$

are equilibria of (3.27). Assumption 3.3(c) implies that (3.30) defines a (n-m)-dimensional equilibrium manifold which is transversal to the dynamic manifolds. The transformation is simply an application of Theorem 3.2.

In the new coordinates (3.22)-(3.23) becomes

$$\begin{aligned} \frac{dy_s}{dt} &= Vf(\gamma(y_s,z)) \\ \varepsilon\,\frac{dz}{dt} &= \varepsilon g_x f(\gamma(y_s,z) + g_x Bk(z+y^*) \end{aligned} \tag{3.31}$$

where $x=\gamma(y_x,z)$ is the inverse transformation to (3.25) which exists because of Assumption 3.3(c).

<u>Corollary 3.5</u> If (i) Assumption 3.3 is satisfied and (ii) the boundary layer system

$$\frac{d\tilde{z}}{d\tau} = g_x\Big|_{x=\gamma(y_s,z)} Bk(\hat{z}+y^*) \tag{3.32}$$

is asymptotically stable, the response of (3.22)-(3.23) is approximated, over a bounded interval [0, T], by

$$\begin{aligned} x(t) &= \gamma(\bar{y}_s(t), \tilde{z}(\tfrac{t}{\varepsilon}) + O(\varepsilon) \\ y(t) &= \tilde{z}(\tfrac{t}{\varepsilon}) + y^* + O(\varepsilon) \end{aligned} \tag{3.33}$$

where $\bar{y}_s(t)$ satisfies the reduced system

$$\frac{dy_s}{dt} = Vf(\gamma(\bar{y}_s, 0)) \quad . \tag{3.34}$$

Proof: Follows immediately from Theorem 3.4 and standard singular perturbation results.

From (3.33) the output differs from the set point $y^*$ by the predominantly fast variable $\tilde{z}+O(\varepsilon)$. Hence, any disturbance that can be modelled as initial condition will appear in the output only over a short initial interval. Note, however, that with the assumptions in Corollary 3.5 the approximation (3.33) is valid only over a bounded time interval [0, T]. Under stronger conditions, which essentially amount to stability requirements on the slow system (3.34), the approximation is valid over unbounded intervals [36] and disturbance rejection is indeed achieved.

Note also that the dynamic manifold defined in (3.29) is linear because we assumed that the input enters linearly through a constant matrix. If in addition the output is linear in x, the equilibrium manifold is linear too and (3.25) is a linear transformation. In this case the boundary layer system (3.32) depends on $\tilde{z}$ only and application of

Theorem 3.4 and Corollary 3.5 is greatly facilitated since transformation (3.25) can be inverted explicitly and stability of (3.32) can be checked more easily. The following example illustrates the discussion above.

Example 3.6 The output of

$$\dot{x}_1 = f_1(x)+u \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.35}$$
$$\dot{x}_2 = f_2(x)+u$$

$$y = x_1^2+x_2 \tag{3.36}$$

is fed back through the high gain law

$$u = \frac{1}{\varepsilon}k(y) = \frac{1}{\varepsilon}(-y-y^3) \quad . \tag{3.37}$$

This system satisfies Assumption 3.3(a), 3.3(b) and

$$V = [1 \quad -1], \qquad y^*=0 \ . \tag{3.38}$$

It also satisfies Assumption 3.3(c) if we restrict the region of validity to $x_1 \geq 0.6$. Transformation (3.25) becomes

$$y_s = x_1-x_2$$
$$z = x_1^2+x_2 \tag{3.39}$$

with inverse

$$x_1 = \frac{-1 \pm \sqrt{1+4\ (y_s+z)}}{2},$$
$$x_2 = \frac{-1 \pm \sqrt{1+4(y_s+z)}}{2} - y_s \tag{3.40}$$

and the transformed system is

$$\frac{dy_s}{dt} = [f_1(x)-f_2(x)]\Big|_{x=\gamma(y_s,z)}$$
$$\varepsilon\frac{dz}{dt} = \varepsilon[2x_1f_1(x)+f_2(x)]\Big|_{x=\gamma(y_s,z)} - [\pm\sqrt{1+4(y_s+z)}\,](z+z^3)\,. \tag{3.41}$$

Since the boundary layer system

$$\frac{d\tilde{z}}{d\tau} = -\sqrt{1+4(y_s+z)}\quad(z+z^3) \tag{3.42}$$

is asymptotically stable in the region of validity, approximation (3.33) holds for large enough gain.

Transformation (3.25) can also be used to analyze disturbance rejection when the disturbance is modelled as an input. Consider

$$\dot{x} = f(x,w(t)) + Bu$$
$$y = g(x) \tag{3.43}$$

under the output feedback

$$u = \frac{1}{\varepsilon}k(y) \tag{3.44}$$

whose difference from (3.22)-(3.23) is that the disturbance w(t) appears as input to the system. Substituting (3.44) into (3.43) we obtain

$$\varepsilon\frac{dx}{dt} = \varepsilon f(x,w(t)) + Bk(g(x)) \tag{3.45}$$

which, under Assumption 3.3, can be transformed to

$$\frac{dy_s}{dt} = Vf(\gamma(y_s,z),\ w(t)) \triangleq \hat{f}(t,y_s,z)$$

$$\varepsilon\frac{dz}{dt} = \varepsilon g_x f(\gamma(y_s,z),w(t)) + g_x Bk(z+y^*) \triangleq \hat{g}(t,y_s,z,\varepsilon) \tag{3.46}$$

using (3.25). In the following corollary we make use of a theorem in [36] which, for convenience is reproduced in Appendix I.

Corollary 3.7 Assume that $w(t)$ is such that $\hat{f},\hat{g}$, satisfy all the conditions of the theorem in [36]. Then under Assumption 3.3 and for high enough gain $\frac{1}{\varepsilon}$ the output of (3.43) remains $0(\varepsilon)$ close to the set point $y^*$ for all $t\in[0,\infty)$.

Proof: Since the boundary layer system of (3.46)

$$\frac{d\tilde{z}}{d\tau} = g_x Bk(\tilde{z}+y^*) \tag{3.47}$$

has $\tilde{z}=0$ as its unique equilibrium and $y(0)=y^*$ implies $z(0)=y(0)-y^*=0$, the fast part of $z$ is $\tilde{z}(\tau)=0\ \forall\ t\in[0,\infty)$. Furthermore, $\bar{z}(t)=0$. Hence,

$$y(t) = z(t) + y^* = y^* + 0(\varepsilon) \tag{3.48}$$

by the theorem in [36].

The practical use of the above corollary may be limited since it assumes that $w(t)$ is known so that the assumptions in [36] can be checked. A much more desirable result would be to establish (3.48) for a class of inputs $w(t)$. Such a result should draw upon the specific way in which the disturbance enters into the problem.

## 3.4 Interconnected Systems

An appealing approach to large scale system analysis and design is to view the system as a collection of dynamic subsystems interacting through static interconnections. The object then is to analyze the stability of [47-49] or design control laws for the system [56] in a decentralized fashion, that is, by testing the stability of the subsystems or designing feedback control using only the subsystem states or outputs. This approach is based on the premise that the connections between subsystems are "weak" compared to the internal connections; hence, qualitative properties and control design can be performed on the subsystem level.

In this section we show that when subsystems have equilibrium and conservation properties, weak connections give rise to two-time-scale behavior. Subsystems are weakly coupled* in the fast time scale but are strongly coupled in the slow one. We start by showing that such cases arise naturally in high gain decentralized output feedback. Consider the interconnected system

$$\begin{aligned} \dot{x}_i &= \hat{f}_i(x_i) + \hat{h}_i(x) + B_i u_i \\ y_i &= \hat{g}_i(x_i) \qquad\qquad i=1,\ldots,\ell \qquad (3.49) \\ x &= [x_1^T \ \ldots \ x_\ell^T]^T \end{aligned}$$

with decentralized output feedback

*We use the word "connection" to mean physical, static interaction and the word "coupling" to imply dynamic interaction.

$$u_i = \frac{1}{\varepsilon} \hat{k}_i(y_i) \qquad i=1,\ldots,\ell \tag{3.50}$$

where $B_i$, $\hat{g}_i$, $\hat{k}_i$ satisfy Assumption 3.3, $i=1,\ldots,\ell$. Substituting (3.50) into (3.49) and rescaling the time $\tau = \frac{t}{\varepsilon}$ we obtain

$$\frac{dx_i}{d\tau} = B_i \hat{k}_i(\hat{g}_i(x_i)) + \varepsilon(\hat{f}_i(x_i)+\hat{h}_i(x)) \quad i=1,\ldots,\ell \tag{3.51}$$

whose subsubtems

$$\frac{dx_i}{d\tau} = B_i \hat{k}_i(\hat{g}_i(x_i)) \qquad i=1,\ldots,\ell \tag{3.52}$$

have equilibrium and conservation properties as argued in section 3.3.

Generalizing (3.51) we consider weakly interconnected systems of the form

$$\frac{dx_i}{d\tau} = f_i(x_i,\varepsilon) + \varepsilon g_i(x,\varepsilon) \qquad i=1,\ldots,\ell \tag{3.53}$$

where $f_i$ is defined on a domain $D_i x[0,\varepsilon_i] \subset R^{n_i} \times R$, $g_i$ is defined on $Dx[0,\varepsilon] \subset R^n \times R$, $x=[x_1^T \ldots x_\ell^T]^T$ and $n = \sum_{i=1}^{\ell} n_i$. Functions $f_i$ and $g_i$ are assumed to be sufficiently smooth. In (3.53) $f_i(x_i,\varepsilon)$ represents the ith isolated subsystem [47] whereas, $\varepsilon g_i(x,\varepsilon)$ represents interconnections with other subsystems. Although the dependence of $f_i$ on $\varepsilon$ may seem superfluous, it is sometimes needed to assure the existence of equilibria of the isolated subsystems; such a case is discussed in the next chapter in relation to dynamic networks. Concerning the isolated subsystems we make the following assumption.

Assumption 3.8 Every isolated subsystem

$$\frac{dx_i}{d\tau} = f_i(x_i,0) \quad , \qquad i=1,\ldots,\ell \tag{3.54}$$

has equilibrium and conservation properties. That is

(i) the set

$$S_i = \{x_i | f_i(x_i,0)=0\} \tag{3.55}$$

is a $\nu_i$-dimensional equilibrium manifold of (3.54), $0 \leq \nu_i \leq n_i$ ; hence, there exists smooth function $\varphi_i : R^{n_i} \to R^{\rho_i}$, $\rho_i = n_i - \nu_i$, such that

$$\varphi_i(x_i)=0 \qquad \Longleftrightarrow \qquad f_i(x_i,0)=0 \tag{3.56}$$

(ii) there exists function $\sigma_i : R^{n_i} \to R^{\nu_i}$ such that

$$F_i = \{x_i | \sigma_i(x_i) = \sigma_i(x_i(0))\} \tag{3.57}$$

is a family of invariant manifolds of (3.54) parametrized on $\sigma_i(x_i(0))$. Moreover, $S_i$ and $F_i$ are nontangent, i.e.

$$\text{rank} \begin{bmatrix} \varphi_{ix} \\ \sigma_{ix} \end{bmatrix} = n \qquad , \qquad \forall\, x_i \in D_i \tag{3.58}$$

where $\varphi_{ix}, \sigma_{ix}$ are the Jacobian matrices of $\varphi_i, \sigma_i$.

<u>Corollary 3.9</u> Under Assumption 3.8 the interconnected system (3.53) is a two-time-scale system and the transformation

$$\begin{aligned} y_i &= \sigma_i(x_i) \\ z_i &= \varphi_i(x_i) \end{aligned} \qquad i=1,\ldots,\ell \tag{3.59}$$

transforms (3.53) into an explicit model with $\nu = \sum_{i=1}^{\ell} \nu_i$ predominantly slow variables y and $\rho = \sum_{i=1}^{\ell} \rho_i$ predominantly fast variables z for which $\bar{z}=0$.

<u>Proof</u>: Isolated subsystems (3.54) form the auxiliary system of (3.53) obtained by setting $\varepsilon=0$.

Defining

$$\varphi(x) = \begin{bmatrix} \varphi_1(x_1) \\ \varphi_2(x_2) \\ \cdot \\ \cdot \\ \cdot \\ \varphi_\ell(x_\ell) \end{bmatrix} , \qquad \sigma(x) = \begin{bmatrix} \sigma_1(x_1) \\ \sigma_2(x_2) \\ \cdot \\ \cdot \\ \cdot \\ \sigma_\ell(x_\ell) \end{bmatrix} \tag{3.60}$$

we obtain equilibrium manifold $\varphi(x)=0$, and dynamic manifolds $\sigma(x(\tau))=\sigma(x(0))$ of the interconnected system satisfying Assumption 3.1. The conclusion is then an application of Theorem 3.2.

Note that the dimension of manifold $S_i$ of each subsystem is allowed to take extreme values 0 and $n_i$. When $\nu_i=0$ (3.54) has at the most isolated equilibria and its dynamic manifold is the whole space $R^{n_i}$; when $\nu_i=n_i$ (3.54) is made of $n_i$ integrators and its equilibrium manifold is $R^{n_i}$. Note also that transformation (3.59) is block diagonal in the sense that $y_i$ and $z_i$ are defined in terms of subsystem state $x_i$ only.

Noting that

$$\begin{aligned} x_i &= \gamma_i(y_i,z_i) \\ x &= \gamma(y,z) = [\gamma(y_1,z_1) \;\; . \;\; . \;\; . \;\; \gamma_\ell(y_\ell,z_\ell)] \end{aligned} \tag{3.61}$$

is the inverse transformation of (3.59) which exists due to (3.58) and rescaling the time $t=\varepsilon\tau$ the transformed model is

$$\frac{dy_i}{dt} = \sigma_{ix}\left[\frac{\partial f_i(\gamma_i(y_i,z_i),\varepsilon)}{\partial\varepsilon} + g_i(\gamma(y,z),\varepsilon)\right] \triangleq F_i(y,z,\varepsilon)$$

$$\varepsilon\frac{dz_i}{dt} = \varphi_{ix}\,[f_i(\gamma_i(y_i,z_i),\varepsilon)+\varepsilon g_i(\gamma(y,z),\varepsilon)] \triangleq G_i(y_i,z_i,\varepsilon)+\varepsilon H_i(y,z,\varepsilon) \tag{3.62}$$

where $\sigma_{ix},\varphi_{ix}$ are the partial derivatives of $\sigma_i,\varphi_i$ with respect to $x_i$. According to Corollary 3.9 the quasi-steady-state is $\bar{z}=0$, and the slow model is

$$\frac{d\bar{y}_i}{dt} = F_i(\bar{y},0,0) \qquad i=1,\ldots,\ell \quad . \tag{3.63}$$

Rescaling back to $\tau$ and setting $\varepsilon=0$ in the second equation of (3.62) we obtain the fast model

$$\frac{d\tilde{z}_i}{d\tau} = G_i(y_i,\tilde{z}_i,o) \qquad i=1,\ldots,\ell \tag{3.64}$$

where $y_i$ appears as a parameter.

Note that $F_i$ is a function of the whole $\bar{y}$ vector whereas, $G_i$ is a function of $\tilde{z}_i$ only. Hence, separation of time scales has resulted in a decomposition in which parts from every subsystem are put together to form a <u>slow core</u> (y-variables) while the rest of each subsystem forms a <u>fast residue</u> ($z_i$-variables). The slow core describes the system-wide dynamics which due to the weak connections between subsystems become significant only in the long run. The fast residues describe the local dynamics which, due to the strong connections within subsystems, are significant in the short run. If further, the fast residues are asymptotically stable the $z_i$ variables reach quickly their quasi-steady-state

equilibrium ($\bar{z}_i=0$); hence, they are weakly coupled with each other since interaction from subsystem to subsystem through the weak connections becomes noticeable only in the slow time scale. (See (3.64)). Figure 3.2 gives a pictorial view of the discussion. This decomposition is very reminiscent of Simon and Ando's reasoning in their classical 1961 paper [46]. We quote:

(1) We can somehow classify all the variables in the economy into a small number of groups;

(2) We can study the interactions within the groups as though the interaction among groups did not exist;

(3) We can define indices representing groups and study the interaction among these indices without regard to the interactions within each group.

Step (1) corresponds in our case to identifying subsystems connected to each other through weak connections. Step (2) corresponds to our fast models (3.64) which are disconnected; we went one step further to remove the slow motion from each subsystem. Step (3) corresponds to the definition of slow variables $y_i$ as "indices" representing subsystems and the study of the system-wide dynamics through the slow core (3.63). In the next chapter where the decomposition is specialized to dynamic networks, the slow "indices" take the meaning of aggregate physical variables.

There is an extensive literature devoted to stability analysis of interconnected systems [47-49 and references therein]. The general plan followed is (i) to regard the large scale system as an interconnection

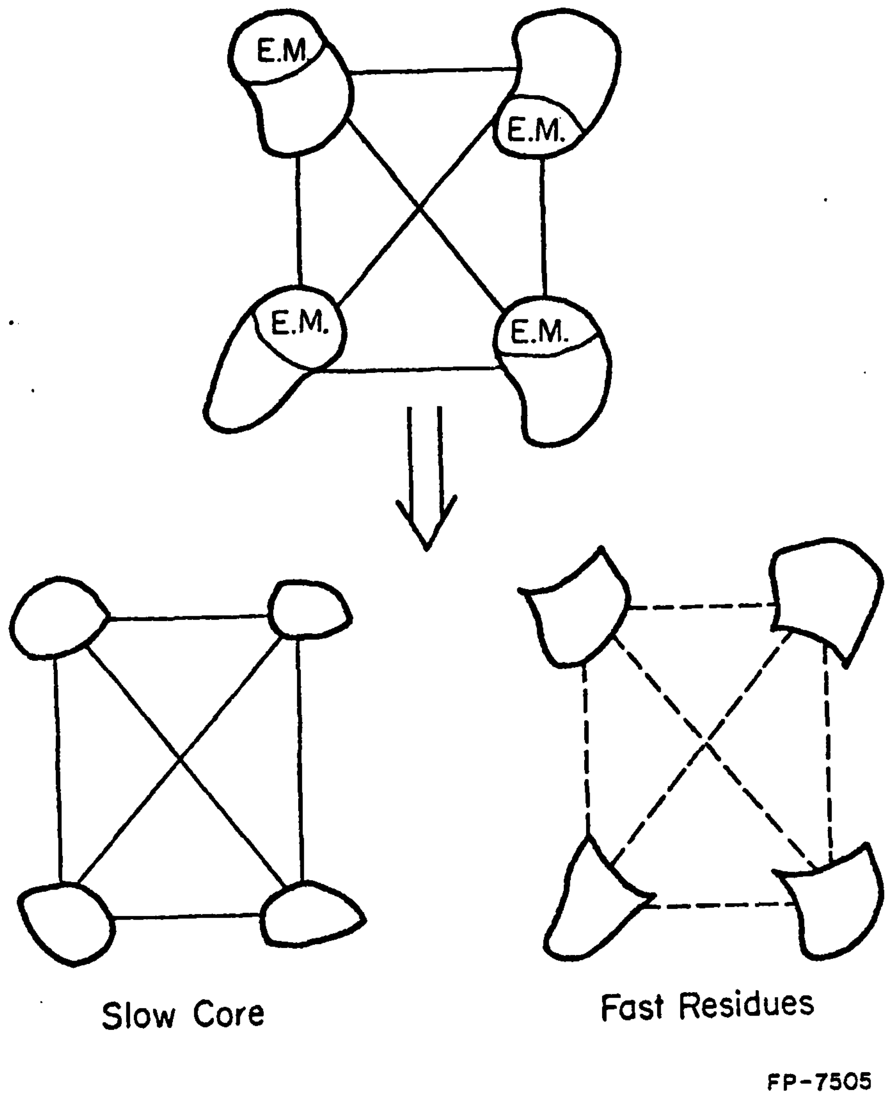


Fig. 3.2 The decomposition into slow core and fast residues.

of isolated subsystems, (ii) to characterize stability properties of isolated subsystems through Lyapunov techniques, (iii) to deduce stability properties of the overall system from stability of subsystems and the nature of interconnections. A basic assumption in [47,48] is that the subsystems have isolated equilibria and the results of [47,48] do not directly apply when the subsystems have nonisolated (i.e., a continuum of) equilibrium points such as in (3.53) with $f_i(x_i,0)$ satisfying Assumption 3.8. We now show that stability criteria analogous to those in [47] can be derived based on recent results [20] on stability of singularly perturbed systems. For convenience, these results are reproduced in Appendix II.

We consider the interconnected system (3.53) when $f_i(x_i,0)$ satisfies Assumption 3.8. Using transformation (3.59) we obtain the system in the explicit singularly perturbed form (3.62). We assume that $y_i \in D_{y_i} \subset R^{\nu_i}$, $z_i \in D_{z_i} \subset R^{\rho_i}$, $i=1,\ldots,\ell$ and that $y=0, z=0$ is the unique equilibrium of (3.62). Moreover the following assumptions are made concerning the slow core, the fast residues and their interactions.

Assumption 3.10

(i) The slow core (3.63) has a Lyapunov function $V: R^{\nu} \to R_+$ such that for all $y \in D_y$

$$[\nabla_y V(y)]^T F(y,0,0) \leq -\alpha_1 \psi^2(y), \quad \alpha_1 > 0$$

where $\psi(y)$ is a scalar valued function of y with $\psi(0)=0$ and $\psi(y)\neq 0$, $y\neq 0$.

(ii) Every isolated fast residue (3.64) has a Lyapunov function $W_i(y_i,z_i): R^{\nu_i} x R^{\rho_i} \to R_+$ such that for all $y_i \in D_{y_i}, z_i \in D_{z_i}$

$$[\nabla_{z_i} W_i(y_i,z_i)]^T G_i(y_i,z_i,0) \leq -\alpha_{2i} \phi_i^2(z_i) \quad \alpha_{2i} > 0.$$

(iii) There exists $\lambda_i > 0$, $i=1,\ldots,\ell$ such that for all $y \in D_y$, $z \in D_z$ the following hold:

(a) $[\nabla_{yi} W_i(y_i,z_i)]^T F_i(y,z,\varepsilon) \leq c_i \sum_{i=1}^{\ell} \lambda_i \phi_i^2(z_i)$

(b) $[\nabla_y V(y)]^T [F(y,z,\varepsilon)-F(y,0,\varepsilon)] \leq \beta \Psi(y) (\Sigma \lambda_i \phi_i^2(z_i))^{1/2}$

(c) $[\nabla_{z_i} W_i(y_i,z_i)]^T [G_i(y_i,z_i,\varepsilon)-G_i(y_i,z_i,0)] \leq \varepsilon k_{1i} \phi_i^2(z_i)$

(d) $[\nabla_{z_i} W_i(y_i,z_i)]^T H_i(y,z,\varepsilon) \leq k_2 \sum_{i=1}^{\ell} \lambda_i \phi_i^2(z_i)$.

Constants $c_i, \beta, k_{1i}, k_2$ are all assumed to be nonnegative numbers.

Theorem 3.11 If Assumption 3.10 is satisfied and $\varepsilon$ is sufficiently small the equilibrium ($y=0,z=0$) of (3.62) is asymptotically stable.

Proof: Let

$$W(y,z) = \sum_{i=1}^{\ell} \lambda_i W_i(y_i,z_i) \tag{3.65}$$

be a tentative Lyapunov function for the boundary layer system, formed by the $z_i$-systems, $i=1,\ldots,\ell$, in (3.64). Then,

$$\begin{aligned} [\nabla_z W(y,z)]^T G(y,z,0) &= \sum_{i=1}^{\ell} \lambda_i \nabla_{z_i} W_i(y_i,z_i) G_i(y_i,z_i,0) \\ &\leq - \sum_{i=1}^{\ell} \lambda_i a_{2i} \phi_i^2(z_i) \leq - \alpha_2 \phi^2(z) \end{aligned} \tag{3.66}$$

where

$$\phi^2(z) = \sum_i \lambda_i \phi_i^2(z_i) \quad , \quad \alpha_2 = \min_i \alpha_{2i} \quad . \tag{3.67}$$

Hence, $V(y), W(y,z)$ satisfy condition (I), (II) of [20]. Condition IIIa of [20] is also satisfied since

$$[\nabla_y W(y,z)]^T F(y,z,\varepsilon) = \sum_i \lambda_i [\nabla_{y_i} W_i(y_i,z_i)]^T F_i(y,z,\varepsilon)$$
$$\leq \sum_i \lambda_i c_i \phi^2(z) = (\sum_i \lambda_i c_i) \phi^2(z) \tag{3.68}$$

whereas condition IIIb is identical to (iiib) in Assumption 3.10. Finally, letting

$$\hat{G}_i(y,z,\varepsilon) = G_i(y_i,z_i,\varepsilon) + \varepsilon H_i(y,z,\varepsilon) \tag{3.69}$$

we have

$$[\nabla_z W(y,z)]^T [\hat{G}(y,z,\varepsilon) - \hat{G}(y,z,0)] =$$
$$\sum_i [\nabla_{z_i} W_i(y_i,z_i)]^T [G_i(y_i,z_i,\varepsilon) - G_i(y_i,z_i,0) + \varepsilon H_i(y,z,\varepsilon)] \leq \tag{3.70}$$
$$\varepsilon \sum_i k_{1i} \phi_i^2(z_i) + \varepsilon k_2 \phi^2(z) \leq \varepsilon K_1 \phi^2(z)$$

where

$$K_1 = \max_i \frac{k_{1i}}{\lambda_i} + k_2 \tag{3.71}$$

and we see that condition IIIc of [20] is also satisfied. The conclusion follows directly from [20].

A similar procedure cannot be applied to the original model (3.53) because the isolated systems possess a continuum of equilibrium points. The basic difference between (3.53) and (3.62) is that in (3.62) all the slow dynamics giving rise to equilibrium manifolds have been relegated to the slow core; consequently, the fast residues no longer have a continuum of equilibrium points. The stability criteria become easy to apply when the transformed models are structured, such as in dynamic networks, the subject of the next chapter.

# CHAPTER 4

# REDUCED ORDER MODELING OF DYNAMIC NETWORKS

## 4.1 Introduction

Much of the equilibrium-conservation reasoning presented in Chapters 2 and 3 was inspired by the study of time scales in power systems and other weakly connected networks. In this class of systems, which is the subject of the present chapter, states and connections have physical meaning and separation of time scales is related to physical laws such as conservation of mass, charge, momentum, etc.

Section 4.2 discusses weakly connected networks with linear storage but nonlinear interconnection elements. The main result is that the transformation that brings the system into the explicit singularly perturbed form is linear. In the new coordinates a slow core describes the system-wide behavior while fast residues describe the local behavior of the network. The slow core turns out to be another "aggregate" network whose states and connections are related in an intuitively appealing way to the states and connections of the original. These results are then specialized to power systems and a five-machine example illustrates the reduction procedure (section 4.3); simulation results are also shown. In section 4.4 we clarify the relation between coherency and localizability [24-25] for LTI systems.

## 4.2 Time Scales in Nonlinear Dynamic Networks

The time-scale separation methodology developed in the previous chapter will now be applied to nonlinear dynamic networks, a class of large systems whose structure facilitates the derivation of reduced models with physical meaning. The dynamic networks we consider are systems comprised of storage elements, capable of storing some physical quantity and interconnection elements capable of transporting this quantity without delay. Examples of dynamic networks include power systems, where angular momentum stored in the generators is transported through transmission lines, R-C networks where charge in the capacitors is transported through resistors, mass-spring systems, etc. The dynamic networks considered first have storage elements with linear characteristics but their interconnections may be nonlinear. Extension to networks with nonlinear storage elements is indicated later in the section. The rates of flow in the interconnections are assumed to be continuously differentiable functions of the potential differences across the interconnections satisfying

$$f_{ik}(x_i-x_k) = - f_{ki}(x_k-x_i) \quad . \tag{4.1}$$

This assumption is equivalent to saying that there are neither sources nor sinks along the interconnection. The dynamics of these systems are then modeled by either the system of first order equations

$$\dot{x}_i = - \frac{1}{m_i} [ \sum_{k \in k_i} f_{ik}(x_i-x_k) - I_i] \tag{4.2}$$

or the system of second order equations

$$\ddot{x}_i = - \frac{1}{m_i} [ \sum_{k \in k_i} f_{ik}(x_i-x_k) - I_i] \tag{4.3}$$

where $x_i, m_i$ the potential and inertia of the ith storage element, $I_i$ the net injection at the ith element, $f_{ik}(\cdot)$ the characteristic of the interconnection between elements i and j and $k_i$ the set of elements to which i is connected. In the remainder of this section we deal with dynamic networks in the form (4.2). In the next section the results will be applied to power systems whose equations are in the form (4.3).

A dynamic network is said to be weakly connected if some interconnections can be expressed as multiples of a small parameter $\varepsilon$. The model of a weakly connected network is then

$$\frac{dx_i}{d\tau} = - \frac{1}{m_i} \Big[ \sum_{k \in K_i} f_{ik}(x_i - x_k) + \varepsilon \sum_{j \in J_i} g_{ij}(x_i - x_j) - I_i(\varepsilon) \Big] \qquad (4.4)$$

where $K_i, J_i$ are index sets representing nodes connected to element i. Constant $I_i$, which depends on and is differentiable with respect to $\varepsilon$, is a net injection (of power or current) at node i. Its dependence on $\varepsilon$ will be discussed later. A fundamental property of weakly connected networks is that neglecting the weak connection terms $\varepsilon g_{ij}$ results in $\nu$ isolated "areas" $\alpha = 1, \ldots, \nu$. Area $\alpha$ contains $n_\alpha$ connected nodes and its equation is obtained by setting $\varepsilon = 0$ in (4.4)

$$\frac{dx_i}{d\tau} = - \frac{1}{m_i} \Big[ \sum_{k \in K_i} f_{ik}(x_i - x_k) - I_i(0) \Big], \quad i \in \alpha, \ \alpha = 1, \ldots, \nu. \qquad (4.5)$$

When the states in each area are ordered consecutively the $n \times \nu$ partition matrix U is

$$U = \text{diag}\ (u_1,\ldots,u_\nu) \tag{4.6}$$

where $u_a$ is an $n_\alpha$ -vector with all elements one.

Assumption 4.1 Each of the $\nu$ areas formed by setting $\varepsilon=0$ in (4.4) has an equilibrium state.

In the case of power systems the above assumption requires that every area, when isolated from the rest of the system, has its own load flow. This will only be possible if the area adjusts its net injections $I_i(\varepsilon)$ so that the power exchanged with other areas is compensated internally. Thus, the injections $I_i(\varepsilon)$ are made functions of the strength of the inter-area connections $\varepsilon$. The choice of the dependence of $I_i$ on $\varepsilon$ and its impact on the accuracy of the reduced models will be discussed later in the section.

In each area $\alpha$ we select a reference node $x_r$, $r\in\alpha$ (*), and form the difference $s_{ir} = x_i^e - x_r^e$ for $r\in\alpha$ and $\forall\ i\in a$, $i\neq r$, where $x_i^e, x_r^e$ are the values of $x_i, x_r$ at an equilibrium $x^e$ of the area model (4.5).

Theorem 4.2 System (4.5) has an equilibrium manifold S described by

$$\varphi_i(x) = x_i - x_r - s_{ir} = 0 \tag{4.7}$$

for $r\in\alpha$; $\forall\ i\in a$, $i\neq r$ and all areas $\alpha=1,\ldots,\nu$. The dynamic manifold $F_{x_o}$ for $x(0)=x_o$ is

$$\sigma_\alpha(x) - \sigma_\alpha(x_o) = 0 \quad , \qquad \alpha=1,\ldots,\nu \tag{4.8}$$

where

$$\sigma_\alpha(x) = \sum_{i\in\alpha} m_i x_i / \sum_{i\in\alpha} m_i \quad . \tag{4.9}$$

(*) Abusing notation, we let $\alpha$ be the index of an area as well as the set of node indices in the area.

Furthermore,

$$y_\alpha(x) = \sigma_\alpha(x) \qquad , \qquad z_i = \varphi_i(x) \tag{4.10}$$

are $\nu$ slow and $\rho=n-\nu$ fast variables satisfying Theorem 3.2.

Proof: Any x satisfying

$$x_i - x_k = x_i^e - x_k^e \qquad \forall \;\; i,k\in\alpha, \quad \alpha=1,\dots,\nu \tag{4.11}$$

is an equilibrium of (4.5). Following a path from node i to node r, these relations can also be written as

$$x_i - x_r = (x_i - x_{i+1}) + (x_{i+1} - x_{i+2}) + \dots + (x_{r-2} - x_{r-1}) + (x_{r-1} - x_r) = x_i^e - x_r^e$$
$$r\in\alpha; \;\; \forall \; i\in\alpha, \;\; i\neq r, \;\; \forall \; \alpha \tag{4.12}$$

which is the expression in (4.7).

Writing (4.5) at an equilibrium

$$0 = -\frac{1}{m_i}\left[\sum_{k\in K_i} f_{ik}(x_i^e - x_k^e) - I_i(0)\right] \quad i\in\alpha, \;\alpha=1,\dots,\nu \tag{4.13}$$

and using (4.1) we obtain

$$\sum_{i\in\alpha} I_i(0) = 0 \qquad , \qquad \alpha=1,\dots,\nu \quad . \tag{4.14}$$

The last relation gives

$$\sum_{i\in\alpha} m_i(dx_i/d\tau) = 0 \qquad , \qquad \alpha=1,\dots,\nu \tag{4.15}$$

where (4.1) was used once more. The dynamic manifold (4.8-4.9) is obtained by integrating and scaling (4.15). Finally the transformation (4.10) is an application of Theorem 3.2.

Note that although the model is nonlinear both the equilibrium manifold (4.7) and the dynamic manifold (4.8) are linear leading to a linear transformation separating the time scales. Manifold S is linear because the right-hand side of (4.5) is a function of a linear combination (the differences) of the states as opposed to being a function of the states individually. Manifolds F are linear because the conservation property is linear. In the case of RC-circuits the conservation property expresses Kirchhoff's current law (KCL) and in the case of power systems the conservation of angular momentum. These physical laws are linear even when some elements of the network have nonlinear characteristics.

To rewrite (4.10) in matrix form we define the difference matrix $G = \text{diag}\ (G_1, \ldots, G_\nu)$ where

$$G_\alpha = \begin{bmatrix} -1 & 1 & 0 & . & . & . & 0 \\ -1 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ -1 & 0 & . & . & . & . & 1 \end{bmatrix} \tag{4.16}$$

is a $(n_\alpha - 1) \times n_\alpha$ matrix with two nonzero elements per row. Ordering the states x in the same area, consecutively, with the reference state first, denoting $M = \text{diag}\ (m_1, \ldots, m_n)$ and $C_a = (U^T M U)^{-1}\ U^T M$ the transformation (4.10) is

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C_a \\ G \end{bmatrix} x - \begin{bmatrix} 0 \\ S \end{bmatrix} \tag{4.17}$$

where s is a $\rho$-vector with components $s_{ir}$. The inverse of (4.17) is

$$x = [U \quad B] \begin{bmatrix} y \\ z+s \end{bmatrix} \tag{4.18}$$

where $B = M^{-1}G^T(GM^{-1}G^T)^{-1}$. Recall from (4.6) that every row of U has one entry 1 and the rest 0, whence,

$$x_i = y_\alpha + b_i(z+s) \qquad \forall \ i \in \alpha, \ \alpha=1,\ldots,\nu \tag{4.19}$$

where $b_i$ is the i row of B. After simple manipulations we obtain the transformed model

$$\frac{dy_\alpha}{d\tau} = -\frac{1}{\sum\limits_{i\in\alpha} m_i} [\varepsilon \sum_{i\in\alpha} \sum_{\substack{j\in J_i \\ \beta=1,\ldots,\nu \\ \beta\neq\alpha}} g_{ij} [(y_\alpha - y_\beta)+(b_i-b_j)(z+s)] - \sum_{i\in\alpha} I_i(\varepsilon) \tag{4.20}$$

$$\begin{aligned} \frac{dz_i}{d\tau} = &-\frac{1}{m_i} [\sum_{\substack{k\in K_i \\ k\neq r}} f_{ik} (z_i - z_k + s_{ir} - s_{kr}) + f_{ir}(z_i + S_{ir}) - I_i(\varepsilon)] \\ &- \frac{1}{m_r} [\sum_{k\in K_r} f_{kr} (z_k + s_{kr}) + I_r(\varepsilon)] \\ &- \varepsilon [\frac{1}{m_i} \sum_{j\in J_i} g_{ij} [(y_\alpha - y_\beta)+(b_i-b_j)(z+s)] \\ &- \frac{1}{m_r} \Sigma g_{rj} [(y_\alpha - y_\beta)+(br-bj)(z+s)]] \quad . \end{aligned} \tag{4.21}$$

In (4.20), (4.21), $i\in\alpha, j\in\beta$ and $b_i$, $b_j$, $b_r$ are the i, j, r rows of B, respectively, $r\in\alpha$. Since from (4.14) $\Sigma I_i(0) = 0$, $\alpha=1,\ldots,\nu$

$$\sum_{i\in\alpha} I_i(\varepsilon) = O(\varepsilon) \qquad , \qquad \alpha=1,\ldots,\nu \tag{4.22}$$

and the right-hand side of (4.20) is $O(\varepsilon)$.

In the original state description (4.4) areas cannot be considered weakly coupled because over a longer period their interaction through weak connections becomes significant. In the transformed description the fast-time area models in z variables are weakly coupled, and the long term area interaction is approximately described by (4.20), that is, the aggregate variables y alone. The fundamental difference between this model and the original is that the decoupled z-equations obtained by setting $\varepsilon=0$ in (4.21), no longer have a continuum of equilibrium points.

The definition of slow coherency as given in [24-28] is based on a modal decomposition and is not directly applicable to nonlinear systems. Since we have shown that the two-time-scale properties remain valid for nonlinear systems, we will use them for the following generalization of the notion of slow coherency.

<u>Slow coherency</u>. States $x_i, x_j$ of (4.4) are said to be slow coherent if $x(0)\in S$ implies $x_i(t)-x_j(t) = \text{const}\ \forall\ t \geq 0$. States $x_i, x_j$ are said to be near slow coherent if there exists a bounded function of time $\zeta(t)$ such that

$$x(0)\in S \Rightarrow x_i(t)-x_j(t) = \text{const.} + \varepsilon\zeta(t),\ \forall\ t \geq 0. \tag{4.23}$$

An area is slow coherent if any two states in the area are near slow-coherent. The following theorem relates weakly coupled areas and slow coherent areas extending the corresponding result in [24-28].

<u>Theorem 4.3</u> If (4.21) satisfies Assumption 1.2 and $\varepsilon$ is sufficiently small, system (4.4) has $\nu$ slow-coherent areas specified by U.

<u>Proof</u>: From Theorem 4.2, if $i,j \in \alpha$

$$x_i - x_j = (x_i - x_r) - (x_j - x_r) = z_i - z_j + s_{ir} - s_{jr}. \tag{4.24}$$

If $x(0) \in S$, $z(0) = \varphi(x(0)) = 0$, which combined with (3.16) and (1.4) implies that $z(t) = 0(\varepsilon)$. Then (4.23) follows from (4.24).

Model (4.20), (4.21) is in the explicit form (1.1). Hence, letting $\varepsilon \to 0$, $I_\alpha = \lim \frac{\Sigma I_i(\varepsilon)}{\varepsilon}$ and using the fact (Theorem 3.2, Theorem 4.2) that $\bar{z} = 0$ the slow model is

$$dy_\alpha/dt = -(1/ \sum_{i\in\alpha} \sum_{\substack{j\in J_i \\ \beta=1,\dots,\nu \\ \beta\neq\alpha}} g_{ij}[(\bar{y}_\alpha - \bar{y}_\beta) + (b_i - b_j)s] \tag{4.25}$$

$$- \sum_{i\in\alpha} \hat{I}_i] \qquad \alpha = 1,\dots,\nu.$$

The fast model is

$$d\tilde{z}_i/d\tau = -(1/m_i)[ \sum_{\substack{k\in K_i \\ k\neq r}} f_{ik}(\tilde{z}_i - \tilde{z}_k + s_{ir} - s_{kr})$$

$$+ f_{ir}(\tilde{z}_i + s_{ir}) - I_i(0)] \tag{4.26}$$

$$-(1/m_r)[ \sum_{k\in K_r} f_{kr}(\tilde{z}_k + s_{kr}) + I_r(0)]$$

$$i \in \alpha, \quad i \neq r \quad \alpha = 1,\dots,\nu.$$

Slow model (4.25) represents an aggregate dynamic network with storage elements

$$m_\alpha = \sum_{i \in \alpha} m_i \qquad \alpha=1,\ldots,\nu \qquad (4.27)$$

net injections

$$I_\alpha = \sum_{i \in \alpha} \hat{I}_i \qquad \alpha=1,\ldots,\nu \qquad (4.28)$$

and interconnection characteristics

$$G_{\alpha\beta}(\bar{y}_\alpha - \bar{y}_\beta) = \sum_{\substack{i \in \alpha \\ j \in \beta}} g_{ij}[\bar{y}_\alpha - \bar{y}_\beta + (b_i - b_j)s]. \qquad (4.29)$$

The aggregate model (4.25) is decoupled from the local models (4.26). Since the sums in (4.26) involve nodes from the same area, the z equations for two different areas are decoupled, that is, the fast models (4.26) are local in the sense that they involve quantities from one area only. Thus, each area uses its local model and at the same time provides the data and receives results from the global model. This multimodeling decomposition helps in formulation of decentralized controls [57].

In (4.4) and in subsequent derivations it was assumed that the dependence of the injections $I_i(\varepsilon)$ on $\varepsilon$ is known. In a realistic situation $\varepsilon$ has a specific value and injections are constant. The dependence on $\varepsilon$ is an asymptotic tool guaranteeing that the isolated areas formed by $\varepsilon \to 0$ have a well defined equilibrium. Therefore, for any function $I(\varepsilon)$

satisfying $\sum_{i\in\alpha} I_i(0) = 0$, $\alpha=1,\ldots,\nu$ Assumption 4.1 holds and the quantities $s_{ir}$ in (4.7) are well defined. However, the freedom in choosing $I(\varepsilon)$ can be utilized to influence the accuracy of reduced models in realistic systems where $\varepsilon$ may not be very small.

Note that for $\varepsilon > 0$, the equilibria of (4.25), (4.26) are generally different from the equilibria of (4.20), (4.21). It has been observed in numerical experiments that the approximation of the time response improves when the equilibrium of the reduced models is closer to the equilibrium of the original (4.20), (4.21). It is desirable to make the two equilibria as close as possible, particularly for oscillatory responses, and if the reduced model (4.25), (4.26) is used for stability analysis. The following corollaries provide guidance in this direction.

Corollary 4.4 Let $x^E$ be the equilibrium of (4.4). The equilibria of (4.25), (4.26) are equal to the equilibria of (4.20), (4.21) if and only if

$$s = Gx^E \quad . \tag{4.30}$$

Proof: First note that by Theorem 3.2 and Theorem 4.2 $\tilde{z} = 0$ is the equilibrium of (4.26), irrespective of the choice of s. From (4.17) the equilibrium of (4.21) corresponding to $x^E$ is

$$z^E = Gx^E - s \tag{4.31}$$

which is made zero by (4.30). Setting $z = 0$ in (4.20) to obtain (4.25) does not alter the equilibrium because $z^E = 0$. The choice (4.30) is unique because (4.31) is linear in s.

Corollary 4.4 shows that there is a unique s for which the equilibria of the exact and approximate systems are equal. Since by definition $s = Gx^e$ and G is a matrix of full rank, (4.30) implies that

$$x^e = x^E \quad . \tag{4.32}$$

The next corollary gives a necessary condition on $I_i(\varepsilon)$ such that (4.32) is satisfied. Boundary nodes are nodes to which interarea connections are attached.

<u>Corollary 4.5</u> Equation (4.32) is satisfied only if

$$I_i(\varepsilon) - I_i(0) = \varepsilon \sum_{j \in J_i} g_{ij} \, (x_i^E - x_j^E) \tag{4.33}$$

that is, the net injection at boundary nodes is adjusted by the interarea flow while it is left unaltered at nonboundary nodes.

<u>Proof</u>: For the equilibrium of (4.4)

$$\sum_{k \in K_i} f_{ik} \, (x_i^E - x_k^E) = - \varepsilon \sum_{j \in J_i} g_{ij} \, (x_i^E - x_j^E) + I_i(\varepsilon) \tag{4.34}$$

and for the equilibrium of (4.5)

$$\sum_{k \in K_i} f_{ik} \, (x_i^e - x_k^e) = I_i(0). \tag{4.35}$$

Hence, (4.33) is necessary for (4.32) to be true.

In cases such as water distribution networks where some storage elements may be nonlinear, the reduction procedure is still applicable after some modification. Assuming that the stored quantity is a strictly

monotonic function of the potential, the dynamics of the network are described by an equation analogous to (4.2) in which $m_i$ is now a function of $x_i$. Equation (4.9) becomes

$$\sigma_\alpha(x) = \frac{\sum_{i\in\alpha} m_i(x_i)x_i}{\sum_{i\in\alpha} m_i(x_i)} \tag{4.36}$$

and the dynamic manifold is no longer linear. The equilibrium manifold, however, is still given by (4.7).

## 4.3 Power System Application

The concepts of area aggregation and slow coherency that emerged from the separation of time scales in dynamic networks (Theorem 4.2) originated as model order reduction techniques in power systems [21-23]. First, a group of coherent generators, that is, generators that "swing together" is identified and then this group is replaced with an equivalent generator. Analytical studies of coherency [58,59] and coherency based aggregation [24-28] were based on linearized versions of the electromechanical model of power systems. In this section we apply the reasoning of Section 4.2 to extend the model simplification approach in [24-28] to the nonlinear electromechanical model, and to more complex models involving flux decay dynamics and voltage regulator.

The well known electromechanical model [60] of multimachine power systems is

$$2H_i\,\ddot{\delta}_i = P_{mi} - P_{ei} \qquad i=1,2,\ldots,\eta \tag{4.37}$$

where $\delta_i$ is the rotor angle of machine i, $H_i$ is its inertia constant, $P_{mi}$ and $P_{ei}$ are its mechanical input power and electrical output power, respectively, and the small damping was ignored [25]. In this model $P_{mi}$ is assumed to be constant and $P_{ei}$ is given by

$$P_{ei} = \sum_{\substack{j=1 \\ j\neq i}}^{n} v_i v_j \; B_{ij} \sin (\delta_i - \delta_j) + v_i^2 \; G_{ii} \tag{4.38}$$

where $v_i$ is the constant voltage "behind the transient reactance," $B_{ij}$ is the (ij)-th entry of the admittance matrix reduced to the machine nodes and $G_{ii}$ represents the load conductance at node i.

Substituting (4.38) into (4.37) we obtain

$$\ddot{\delta}_i = - \frac{1}{2H_i} [ \sum_{\substack{j=1 \\ j\neq i}}^{n} v_i v_j \; B_{ij} \sin (\delta_i - \delta_j) - (P_{mi} - v_i^2 \; G_{ii})] \tag{4.39}$$

which is in the form of (4.3) with $m_i = 2H_i$, $x_i = \delta_i$,

$$f_{ij} \; (x_i - x_j) = v_i v_j \; B_{ij} \sin (\delta_i - \delta_j) \tag{4.40}$$

$$I_i = P_{mi} - v_i^2 \; G_{ii} \quad . \tag{4.41}$$

Multimachine power systems are often comprised of groups of tightly connected machines with weak connections joining the groups. Assuming that weak connections are known system (4.39) takes the form (4.4) for which Theorem 4.2 gives equilibrium and dynamic manifolds and defines slow and fast variables. Note, however, that since damping was neglected the response of (4.39) is purely oscillatory and the separation of time scales

is understood in the sense of separating low frequency from high frequency oscillations [61]. Using transformation (4.10) we can arrive at expressions similar to (4.20)-(4.21) from which the slow core and the fast residues are defined as in (4.25)-(4.26). The reduction procedure in power systems and some physical interpretations of the reduced models are illustrated through the following five-machine example.

In the power system of Fig. 4.1, $H_i = 0.5$, $v_{i=1}$, $i=1,\ldots,5$ and $B_{12}=B_{23}=B_{45}=1$, $B_{34}=B_{25}=B_{15}=B_{14}=0.1$. The net injections $I_i$ and the resulting steady-state angles (in radians) are given in columns 1 and 2 of Table 4.1.

Table 4.1. Bus angles for five-machine power example

| 1 | 2 | 3 |
|---|---|---|
| $I_1 = -0.28$ | $\delta_1 = 0$ | $\delta_1 = 0$ |
| $I_2 = -0.077$ | $\delta_2 = 0.171$ | $\delta_2 = 0.215$ |
| $I_3 = 0.186$ | $\delta_3 = 0.391$ | $\delta_3 = 0.458$ |
| $I_4 = 0.362$ | $\delta_4 = 0.723$ | $\delta_4 = 1.042$ |
| $I_5 = -0.191$ | $\delta_5 = 0.456$ | $\delta_5 = 0.730$ |

Note that since admittances $B_{34}$, $B_{25}$, $B_{15}$, $B_{14}$ are much smaller than the rest, the system is divided into two weakly connected areas $\alpha = \{1,2,3\}$, $\beta = \{4,5\}$. Suppose now that line $B_{14}$ is tripped and we want to simulate the resulting oscillations using reduced models (4.25)-(4.26). The post fault load flow (shown in column 3 of Table 4.1) gives

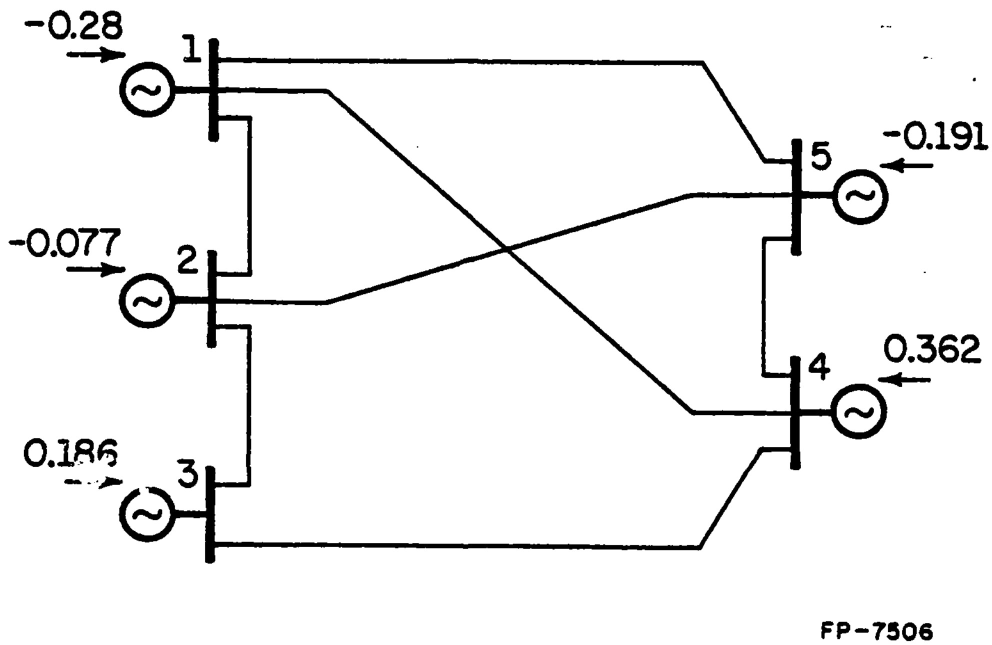


Fig. 4.1 Five-machine power system example.

$$s_{12} = \delta_1^e - \delta_2^e = -\ 0.215$$
$$s_{32} = \delta_3^e - \delta_2^e = 0.243 \tag{4.42}$$
$$s_{45} = \delta_4^e - \delta_5^e = 0.312.$$

Defining new variables

$$y_1 = \frac{\delta_1+\delta_2+\delta_3}{3} \quad , \quad y_2 = \frac{\delta_4+\delta_5}{2}$$
$$z_1 = \delta_1-\delta_2-s_{12} \quad , \quad z_2 = \delta_3-\delta_2-s_{32} \quad , \quad z_3 = \delta_4-\delta_5-s_{45} \tag{4.43}$$

and letting $\varepsilon \to 0$ we obtain the slow model

$$\begin{aligned}\ddot{\bar{y}}_1 &= -\ 0.033 \sin (\bar{y}_1-\bar{y}_2-0.068) - 0.033 \sin (\bar{y}_1-\bar{y}_2+0.147) \\ &\qquad - 0.033 \sin (\bar{y}_1-\bar{y}_2+0.078)-0.057 \\ \ddot{\bar{y}}_2 &= -\ 0.05 \sin (\bar{y}_2-\bar{y}_1-0.078) - 0.05 \sin (\bar{y}_2-\bar{y}_1-0.147) \\ &\qquad - 0.05 \sin (\bar{y}_2-\bar{y}_1+0.068)+0.086\end{aligned} \tag{4.44}$$

and the fast model

$$\begin{aligned}\ddot{\tilde{z}}_1 &= -\ 2 \sin (\tilde{z}_1-0.215) - \sin (\tilde{z}_2+0.243) - 0.185 \\ \ddot{\tilde{z}}_2 &= -\ 2 \sin (\tilde{z}_2+0.243) - \sin (\tilde{z}_1-0.215) + 0.269\end{aligned} \tag{4.45}$$

$$\ddot{\tilde{z}}_3 = -\ 2 \sin (\tilde{z}_3+0.312) + 0.614 \quad . \tag{4.46}$$

Note that (4.44) is decoupled from (4.45), (4.46). The aggregate model (4.44) represents the oscillations of the aggregate angles $y_1, y_2$ against each other, whereas, the local models (4.45), (4.46) represent the

intermachine oscillations in areas $\alpha,\beta$, respectively. Figures 4.2-4.5 show simulation curves with initial conditions equal to the prefault equilibrium. Figures 4.2, 4.3 show exact (solid lines) and approximate (dotted lines) responses of angles $\delta_1,\delta_4$, whereas, Figures 4.4, 4.5 show exact (dotted lines) and approximate (solid lines) responses of the transformed variables $y_1,z_2$. Note that generator angles are mixed variables, whereas, $y_1$ is predominantly slow and $z_2$ is predominantly fast.

We now turn to more complex models of power systems and again investigate the effect of weak connections on the time scale behavior of the system. The model we employ is basically the one in [62] with a slight simplification; we do not include the fictitious quadrature axis coil g which is meant to model eddy currents in the rotor. With this simplification the model is

$$\dot{\delta}_i = 377(w_i-1) \tag{4.47a}$$

$$2H_i\dot{\omega}_i = \frac{P_{mi}}{w_i} - e'_{qi}\ i_{qi} - D_i(\omega_i-1) \tag{4.47b}$$

$$\dot{e}'_{qi} = \frac{1}{T'_{doi}}\ [e'_{qi} - (x_{di}-x'_{di})\ i_{di}+E_{fdi}] \tag{4.47c}$$

$$\dot{R}_{fi} = \frac{1}{T_{Fi}}\ (-R_{Fi} + \frac{K_F}{T_F}\ E_{fdi}) \tag{4.47d}$$

$$\dot{E}_{fdi} = \frac{1}{T_{Ei}}\ [-(K_E + S_E\ (E_{fdi}))\ E_{fdi} + V_{Ri}] \tag{4.47e}$$

$$i_{di} = -\sum_j B_{ij}\ e'_{qj}\ \cos\ (\delta_i-\delta_j) \tag{4.48a}$$

$$i_{qi} = \sum_j B_{ij}\ e'_{qj}\ \sin\ (\delta_i-\delta_j) \tag{4.48b}$$

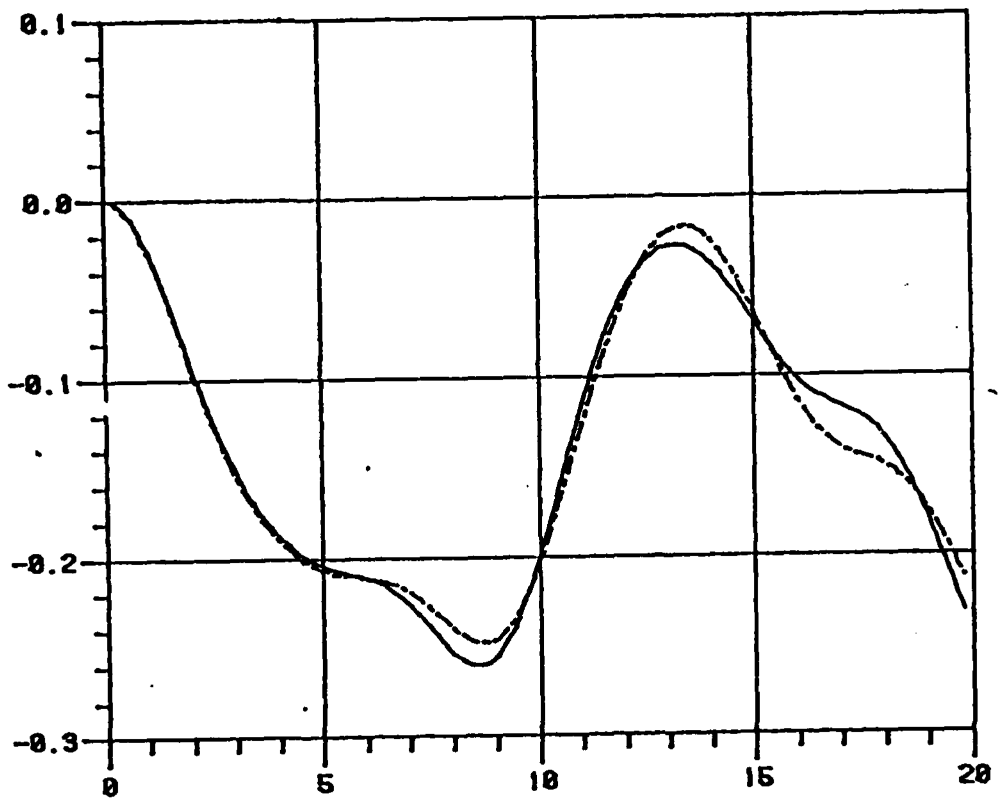


Fig. 4.2 Exact (solid line) and approximate (dotted line) response of $\delta_1$.

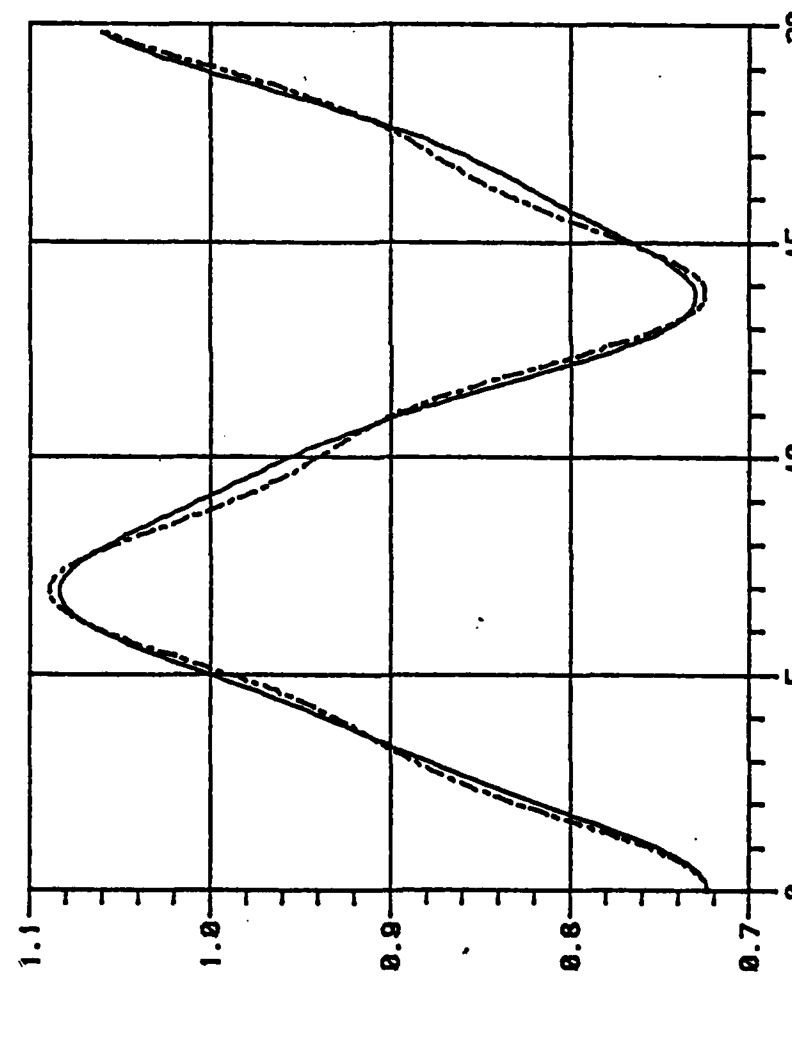


Fig. 4.3 Exact (solid line) and approximate (dotted line) responses of $\delta_4$.

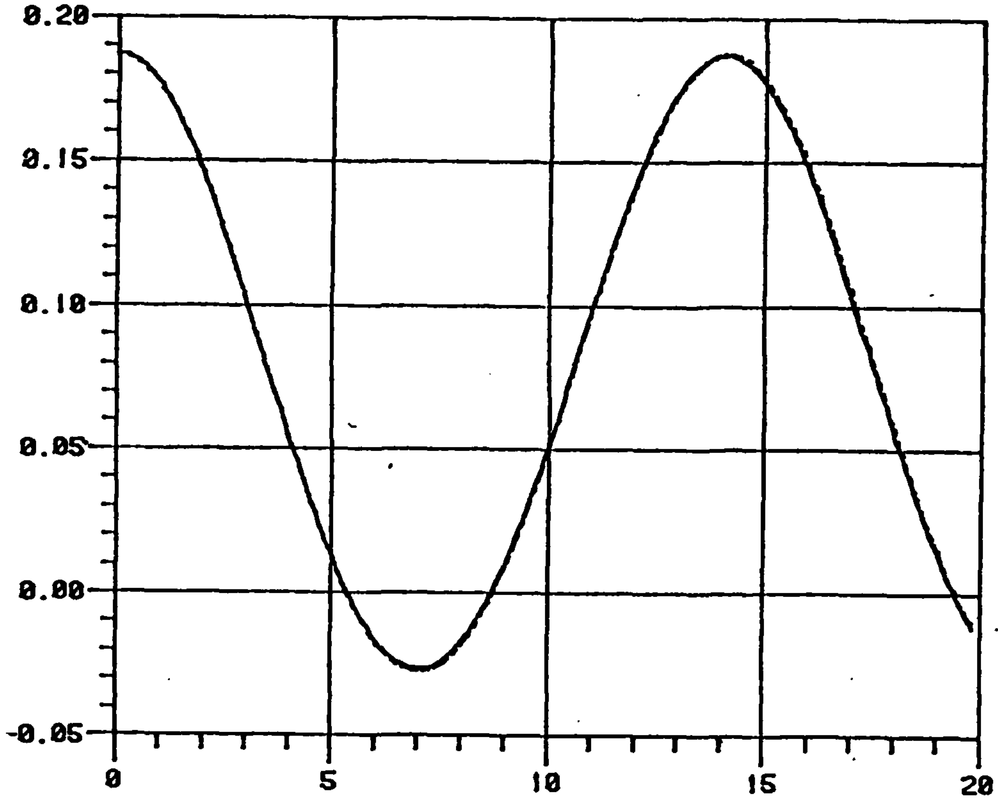


Fig. 4.4 Exact (dotted line) and approximate (solid line) responses of $y_1$.

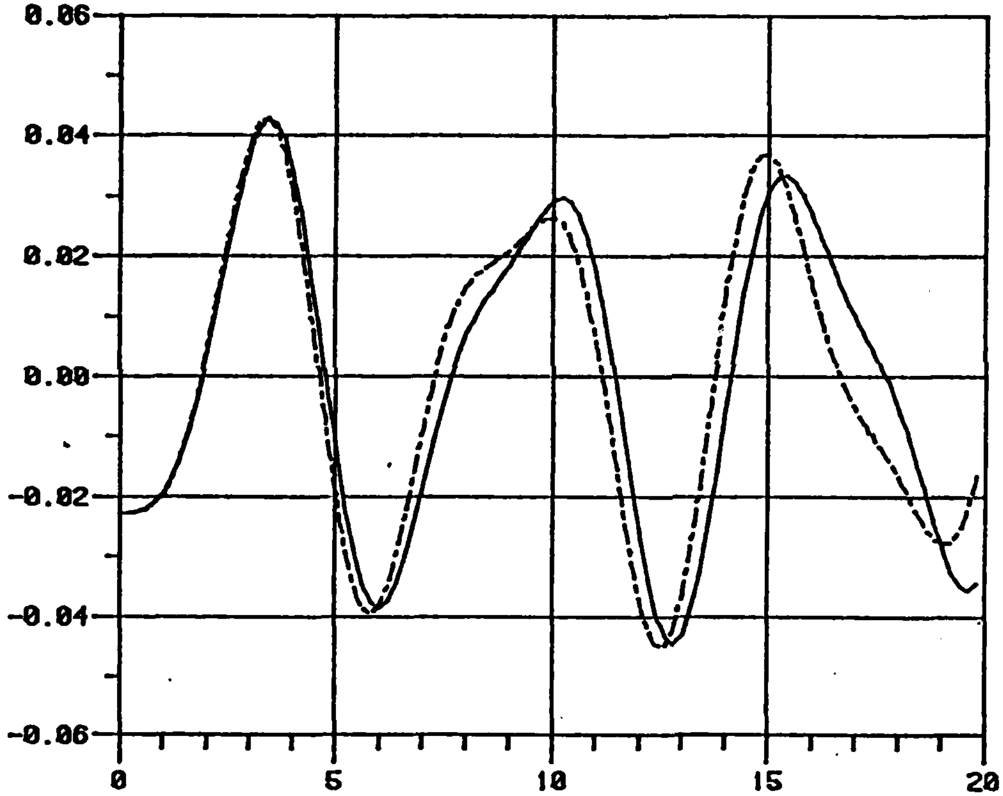


Fig. 4.5 Exact (dotted line) and approximate (solid line) responses of $z_2$.

$$V_{ti} = \sqrt{V_{di}^2 + V_{qi}^2}\ , \quad V_{di} = e'_{di} + x' i_{qi}\ , \quad V_{qi} = e'_{qi} - x' i_{di} \tag{4.49}$$

where state $e'_q$ is proportional to the field flux, equations (4.47d-f) model the voltage regulator-exciter system and (4.48), (4.49) give the interaction of the generators through the transmission network.

If the power system is made of $\nu$ weakly connected areas, (4.48a), (4.48b) are written as

$$i_{di} = -\sum_{j\in\alpha} B_{ij}\, e'_{qj} \cos(\delta_i - \delta_j) - \varepsilon \sum_{k\notin\alpha} B_{ik}\, e'_{qk} \cos(\delta_i - \delta_k) \tag{4.50a}$$

$$i_{qi} = \sum_{j\in\alpha} B_{ij}\, e'_{qj} \sin(\delta_i - \delta_j) + \varepsilon \sum_{k\notin\alpha} B_{ik}\, e'_{qk} \sin(\delta_i - \delta_k) \tag{4.50b}$$

where $i\in\alpha$, $\alpha=1,\ldots,\nu$. We now make the important observation that (i) machines interact solely through currents $i_d, i_q$ and (ii) $i_d, i_q$ are functions of the differences $\delta_i - \delta_j$ of angles, not of angles individually. Hence, letting $\varepsilon=0$ in (4.50) and setting the right-hand side of (4.47) equal to zero we see that if an assumption analogous to 4.1 is met, points satisfying

$$\begin{aligned}
\delta_i - \delta_r - s_{ir} &= 0 \\
\omega_i &= 1 \\
e'_{qi} &= e'_{qie} \\
R_{fi} &= R_{fie} \\
E_{fdi} &= E_{fdie} \\
V_{Ri} &= V_{Rie}
\end{aligned} \tag{4.51}$$

where subscript e denotes equilibrium and $r \in \alpha$, $i \in \alpha$, $i \neq r$, $\alpha = 1,\ldots,\nu$ are equilibrium points of (4.47). That is, (4.47) at $\varepsilon = 0$ has a $\nu$-dimensional equilibrium manifold described by (4.51). An argument similar to the one in Theorem 4.2 shows that

$$\sigma(\delta,\omega) = \sum_{i \in \alpha} \frac{D_i}{377}\,\delta_i + \sum_{i \in \alpha} 2H_i\omega_i = \sigma(\delta(0),\omega(0)) \tag{4.52}$$

for $\alpha = 1,\ldots,\nu$ defines the family of dynamic manifolds of (4.47). In all realistic cases $D_i \ll 377$ so that we can ignore the first term in (4.52). Note that although we started with a higher-dimensional model of the power system the equilibrium and conservation relations (4.51), (4.52) involving $\delta,\omega$ variables are identical to the ones obtained by working with the electromechanical model. Consequently, we obtain an electromechanical slow model involving the aggregate variables $\delta_\alpha = \left(\sum_{i \in \alpha} 2H_i\,\delta_i\right) / \sum_{i \in \alpha} 2H_i$, $\omega_\alpha = \dot{\delta}_\alpha$.

## 4.4 Coherency and Localizability

As shown in Section 4.2 weak connections in a dynamic network give rise to slow-coherent groups of states which are described by local models. We now restrict ourselves to linear time invariant systems and investigate the relation between coherency and localizability when weak connections are not present. This discussion clarifies the presentation in [24,25] where the two notions were essentially treated as equivalent. For the sake of completeness we repeat here the definition of coherency of [24,25].

<u>Definition 4.6</u> Let $\dot{x}=Ax$ be a LTI system, $\sigma$ be a subspectrum of A and $V_\sigma$ be the corresponding eigenspace. Then states $x_i$ and $x_j$ are said to be coherent with respect to $\sigma$ if and only if, $x(0) \in V_\sigma$ implies that

$$x_i(t) = x_j(t) \qquad , \qquad \forall \ t \geq 0 \ . \tag{4.53}$$

A group of states is said to be a $\sigma$-coherent group if any two states from the group are coherent with respect to $\sigma$.

Suppose now that $n_\alpha$ states of a system are considered to belong to a group $\alpha$. The criterion for such grouping can be geographic proximity, accessibility to remote sensing or similar. In an attempt to describe the "local" behavior in the group we use $n_\alpha-1$ differences $x_i-x_k$, where i,k belong to the $\alpha$ set of indices. Typically, we fix index k as the local reference and take all $i \neq k$ in the group $\alpha$ to form the differences. We then investigate under what conditions the local variables

$$z = [G \ \vdots \ 0] \ x = G_o \ x \tag{4.54}$$

where

$$G = \begin{array}{c} \xleftarrow{\hspace{2em}} n_\alpha \xrightarrow{\hspace{2em}} \\ \begin{bmatrix} -1 & 1 & 0 & . & . & . & 0 \\ -1 & 0 & 1 & . & . & . & 0 \\ . & . & . & & & & . \\ -1 & 0 & . & . & . & . & 1 \end{bmatrix} \end{array} \quad \updownarrow n_\alpha - 1 \tag{4.55}$$

are independent of the rest of the system.

<u>Definition 4.7</u> Group $\alpha$ is said to be <u>localizable</u> if there is an $A_\alpha$ such that [*]

$$\dot{z} = A_\alpha z \tag{4.56}$$

<u>Lemma 4.8</u> A group of states is localizable if and only if it is a $\sigma$-coherent group and

$$n_\sigma = n - n_\alpha + 1 \tag{4.57}$$

where $n_\sigma$ is the number of modes in $\sigma$.

<u>Proof</u>: If the group is localizable z can be decoupled from system implying that $n-n_\alpha+1$ modes are unobservable from z. If $V_\sigma$ is a basis of the eigenspace corresponding to these modes

$$G_o V_\sigma = 0 \tag{4.58}$$

which implies that rows of $V_\sigma$ corresponding to states in the group are equal. Hence, it is a coherent group. Conversely, the rows of $V_\sigma$ corresponding to a coherent group are equal implying that $n_\sigma$ modes are unobservable. When $n_\sigma$ satisfies (4.57) the number of observable modes is $n_\alpha-1$ which equals the dimension of z and the group is localizable.

Note that for $V_\sigma$ to be full rank $n_\sigma$ has to satisfy

$$n_\sigma \leq n - n_\alpha + 1 \tag{4.59}$$

---

[*] The notion of localizability is identical to aggregability with respect to matrix $G_o$ of (4.54). To avoid confusion we reserve the latter term for "area" aggregation.

Hence, for the coherent group to be localizable $n_\sigma$ is required to take its maximum permissible value. Note also that the smaller the group the larger $n_\sigma$ and the harder it is to satisfy the localizability conditions. On the other hand we do not benefit much by localizing a large group. Finally, note that the only modes observable from the local variables of a localizable group are the complementary modes $\sigma^c$, henceforth, called <u>local modes</u>.

When a system is divided into more groups of states the localizability conditions can be applied independently to each group. As an example, let the modal matrix of a 5-state system be

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \end{array} \\ \begin{bmatrix} a & b & c & * & * \\ a & b & c & * & * \\ a & b & c & * & * \\ * & n & k & \ell & m \\ * & n & k & \ell & m \end{bmatrix} \end{array} \begin{array}{l} \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \alpha \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \beta \end{array} \tag{4.60}$$

where the stars can be any numbers such that the matrix is nonsingular. Group $\alpha = \{x_1, x_2, x_3\}$ is a $\sigma_\alpha$-coherent group where $\sigma_\alpha = \{\lambda_1, \lambda_2, \lambda_3\}$ and satisfies (4.57). Hence, it is localizable and its local modes are $\sigma_\alpha^c = \{\lambda_4, \lambda_5\}$. Likewise group $\beta = \{x_4, x_5\}$ is $\sigma_\beta = \{\lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ satisfies (4.57) and it is localizable with local modes $\sigma_\beta^c = \{\lambda_1\}$. When, as in the above example both the groups and the local modes are disjoint, the system is called <u>multi-localizable</u>.

As it is clear from (4.60) the multi-localizability conditions are very stringent. Note that in a multi-localizable system, each set of local variables decouples from the rest of the system and observes only the local modes. A less stringent requirement is that the local variables from the different groups decouple from the system as a single set.

Definition 4.9 Let the states of $\dot{x}=A_x$ be divided into r disjoint groups, each consisting of two or more states, and s states not assigned to any group. Then the system is called decomposable if the local variables

$$z = \left[\begin{array}{ccc|c} G_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & G_r & \end{array}\right] x = G_T\, x \tag{4.61}$$

decouple from the system that is if there is a matrix $A_\ell$ such that

$$\dot{z} = A_\ell\, z \tag{4.62}$$

where

$$G_\nu = \begin{array}{c c} & \xleftarrow{\qquad n_\nu \qquad} \\ n_\nu - 1 \updownarrow & \begin{bmatrix} -1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ -1 & 0 & \cdot & \cdot & & \cdot & 1 \end{bmatrix} \end{array} \tag{4.63}$$

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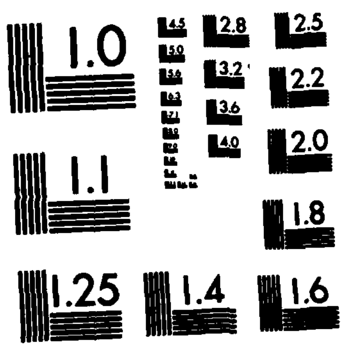
1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0
5.6

$\eta_\nu$ is the number of states in group $\nu$ and the s single states are the last entries of x.

The following lemma establishes the relationship between decomposability and coherency.

Lemma 4.10 A system is decomposable if and only if each of its groups is coherent with respect to the same set of modes $\sigma_a$ and

$$r + s = n_{\sigma a} \tag{4.64}$$

where $n_{\sigma a}$ the number of modes in $\sigma_a$.

Proof: If a system is decomposable, only n-s-r modes are observable from the same number of local variables z. Let $\sigma_a$ be the set of r+s unobservable modes and V a basis for the corresponding eigenspace. Then

$$G_T V = 0 \tag{4.65}$$

which implies that each of the r groups is a coherent group with respect to $\sigma_a$. Conversely if all the groups are coherent with respect to $\sigma_a$ modes are unobservable from z. If further the number of the z variables n-s-r equals the number of observable modes $n-n_{\sigma a}$, that is (4.64) is satisfied, the system is decomposable.

As an illustration consider again groups $\alpha$ and $\beta$ in (4.60). Both $\alpha$ and $\beta$ are coherent with respect to $\sigma_a = \{\lambda_2, \lambda_3\} = \sigma_\alpha \cap \sigma_\beta$ and the two modes $\sigma_a$ equal in number the two areas. Hence, the system in (4.60) is decomposable.

# CHAPTER 5

# SUGGESTIONS AND CONCLUSIONS

## 5.1 Suggestions for Further Research

The ideas in this thesis can be extended in several directions. In Chapters 2 and 3 we showed that equilibrium and conservation properties of an auxiliary system imply multi-time-scale behavior. However, we did not investigate the relation between the two properties. Does the existence of one property imply the other? And under what conditions? It is clear that in Linear Time Invariant systems with simple structure of $\mathcal{N}(A_o)$ (Equations (2.11)-(2.13)) the two properties are equivalent. We feel that the existence of an equilibrium manifold implies conservation properties in a wide class of nonlinear systems. This issue and the one of systematic procedures for finding equilibrium and dynamic manifolds deserve further investigation. A look at the decomposition in [63] and the differential geometry techniques used therein should prove useful. Time scales in discrete-time systems is a rather neglected topic. Does the coordinate-free characterization carry over to this class of systems? And how are time scales related to the sampling period?

We have dealt mostly with time scales of free systems. On the other hand, high gain control is known to change the time-scale behavior (Section 3.3) of systems. In Section 2.5 we have given conditions under which nonexplicit controlled LTI models can be transformed to explicit controlled models. Similar results for nonlinear systems would be desirable.

This issue is related to extension of the high gain results of Section 3.3 to wider classes of systems. A rather easy extension would be to include dependence of matrix B on x.

The decomposition into a slow core and fast residues seems promising in decentralized and hierarchical control design along the lines of [57]. When dealing with physical systems such as power systems, prospects for implementation of such designs should be a consideration.

In terms of practical significance the time scale decomposition of dynamic networks in Sections 4.2,4.3 seems to be the most promising. Stability tests by decomposition methods have been used in power systems [64] but they usually give conservative results. We feel that the decomposition into slow core and fast residues takes advantage of the structure of the system (weakly connected areas) and it is likely to provide practical results. Moreover, it can furnish information on the type of instability, that is intermachine of interarea instability. It would also be interesting to study time-scale separation and stability questions using more complex generator models and an unreduced network.

## 5.2 Conclusions

Singular perturbations have been related to equilibrium and conservation properties of an auxiliary system. Besides providing a coordinate-free characterization of singularly perturbed systems, these properties have been used in definition of new predominantly slow and predominantly fast coordinates. In the new coordinates an extensive amount of literature provides simplified models, asymptotic calculations, two-stage designs and stability tests.

Results on high gain feedback control have been extended to a class of systems much larger than LTI systems. Disturbance rejection behavior has been shown to be similar to the one in the LTI case.

The relation between weak connections and time scales in a class of interconnected systems has been established. Weak connections combined with equilibrium and dynamic manifolds of the subsystems give rise to multi-time-scale behavior. Separation of the time scales results in a slow core describing the system-wide dynamics and a set of fast residues describing the local dynamics. In the new representation recent stability results can be applied to give decentralized stability tests.

When the interconnected system has the added structure of a dynamic network the slow core and the fast residues acquire physical significance and the definition of slow and fast variables is related to physical laws. The linearity of these laws makes the transformation separating the time scales linear. This transformation is the area aggregation-slow coherency one, developed for linearized models of power systems.

## APPENDIX I. SINGULAR PERTURBATION ON THE INFINITE HORIZON

Under consideration is the system

$$\text{(P)} \qquad \begin{aligned} \frac{dy}{dt} &= f(t,y,z,\varepsilon) & \qquad y(t_o) &= y_o \\ \varepsilon\frac{dz}{dt} &= g(t,y,z,\varepsilon) & \qquad z(t_o) &= z_o \end{aligned}$$

with degenerate system

$$\text{(D)} \qquad \frac{dy}{dt} = f(t,y,0,0) \quad , \quad y(t_o) = y_o$$

and boundary layer system

$$\text{(BL)} \qquad \frac{dz}{d\tau} = g(\alpha,\beta,z,0)$$

where $(\alpha,\beta)$ are treated as parameters. In (P) x,f are k-vectors, g,y are j-vectors and, without loss of generality it is assumed that $g(t,y,0,0) = 0$ for all t,y.

Let $|x| = \Sigma\, |x_i|$ be the norm of x, let $I = [0,\infty]$, $S_R = \{(y,z) \varepsilon E^{k+j}\colon |y| + |z| \leq R\}$ and let $S_{R|y}, S_{R|z}$ represent the restrictions of $S_R$ to $E^k$ and $E^j$.

The following assumptions are made about (P), (D), (BL).

(I) System (P) has a solution $y = y(t)$, $z = z(t)$ that exists for $t_o \leq t < \infty$.

(II) $f, g, f_y, f_z, g_t, g_y, g_z \in C$ where $f_x$ denotes the matrix $\dfrac{\partial f_i}{\partial y_\ell}$, $i,\ell = 1,\dots,k$.

(III) Function f is continuous at $z=0, \varepsilon=0$ uniformly in $(t,y) \in I \times S_R|_y$ and $f(t,y,0,0)$, $f_y(t,y,0,0)$ are bounded on $I \times S_R|_y$.

(IV) Function g is continuous at $\varepsilon=0$ uniformly in $(t,y,z) \in I \times S_R$, and $g(t,y,z,0)$ and its derivatives with respect to t and the components of y, z are bounded on $I \times S_R$.

To simplify notation let $\mathcal{K}$ be the class of all continuous, strictly increasing, real valued functions $d(r)$, $0 \leq r$ with $d(0) = 0$; and let $\mathcal{L}$ be the class of all nonnegative, strictly decreasing, continuous, real-valued functions $\sigma(s)$, $0 \leq s < \infty$ for which $\sigma(s) \to 0$ as $s \to \infty$.

(V) The zero solution of D is uniform-asymptotically stable. That is, if $x = \Phi(t,t_o,y_o)$ is the solution of (D), $\exists\, d \in \mathcal{L}$ such that

$$\Phi(t,t_o,y_o) \leq d(|y_o|)\, \sigma(t-t_o) \text{ for } |y_o| \leq R,\ 0 \leq t_o \leq t < \infty.$$

(VI) The zero solution of (BL) is uniform-asymptotically stable uniformly in the parameter $(\alpha,\beta) \in I \times S_R|_y$. That is, if $y = \Psi(s,z_o,\alpha,\beta)$ is the solution of BL, $\exists\, e \in \mathcal{K}, \rho \in \mathcal{L}$, such that

$$\Psi(s,z_o,\alpha,\beta) \leq e(|z_o|)\, \rho(s)$$

for all $0 \leq s < \infty$, $|y_o| \leq R$ and $(\alpha,\beta) \in I \times S_R|_y$.

Then the following Theorem is true [36].

<u>Theorem</u> [36] Let conditions (I) through (VI) be satisfied. Then for sufficiently small $|y_o| + |z_o|$ and $\varepsilon$ the solution of the perturbed system (P) exists for $t_o \leq t < \infty$, and this solution converges to the solution of the degenerate system (D) as $\varepsilon \to 0^+$ uniformly on all closed subsets of $t_o < t < \infty$.

## APPENDIX II. A STABILITY THEOREM

Consider

$$
\text{(P)} \qquad
\begin{aligned}
\dot{y} &= f(y,z,\varepsilon) \qquad & y \in B_y \subset R^n \\
\varepsilon\dot{z} &= g(y,z,\varepsilon) \qquad & z \in B_z \subset R^m
\end{aligned}
$$

where $B_y, B_z$ denote closed spheres centered around $y=0$, $z=0$. Assume that $y=0$, $z=0$ is the unique equilibrium of (P) in $B_y, B_z$ and that $g(y,0,0) = 0$, for all $y \in B_y$. The reduced system of (P) is

$$
\text{(R)} \qquad \dot{y} = f(y,0,0) \triangleq f_r(y)
$$

and its boundary layer is

$$
\text{(BL)} \qquad \frac{dz}{d\tau} = g(y,z(\tau),0).
$$

Let the following assumptions be satisfied.

(I) Reduced system (R) has a Lyapunov function $V: R^n \to R$, such that for all $y \in B_y$,

$$
[\nabla_y V(y)]^T f_r(y) \le -\alpha_1 \Psi^2(y) \quad , \quad \alpha_1 > 0
$$

where $\Psi(y)$ is a scalar-value d function of $y$ with $\Psi(0) = 0$, $\Psi(y) \neq 0$ if $y \neq 0$.

(II) Boundary-layer system (BL) has a Lyapunov function $W(y,z): R^n \times R^m \to R$ such that for all $y \in B_y$, $z \in B_z$

$$[\nabla_z W(y,z)]^T g(y,z,0) \leq -\alpha_2 \phi^2(z) \qquad \alpha_2 > 0$$

where $\phi(z)$ is scalar valued and $\phi(0)=0$, $\phi(z)\neq 0$ if $z\neq 0$.

(III) The following inequalities hold for all $y\in B_y$, $z\in B_z$

(a) $[\nabla_y W(y,z)]^T f(y,z) \leq C_1 \phi^2(z) + C_2 \phi(z) \Psi(y)$

(b) $[\nabla_y V(y)]^T [f(y,z) - f(y,0)] \leq \beta_1 \Psi(y) \phi(z)$

(c) $[\nabla_z W(y,z)]^T [g(y,z,\varepsilon) - g(y,z,0)] \leq \varepsilon K_1 \phi^2(z) + \varepsilon K_2 \Psi(y) \phi(z)$

<u>Theorem</u> [20] If conditions (I)-(III) are true, the origin x=0,y=0 is an asymptotically stable equilibrium point of (P).

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# VITA

George Michael Peponides was born in Nicosia, Cyprus on May 16, 1953. He received the Diploma in Physics from the University of Athens, Greece in 1976. While at the University of Athens he was supported by a scholarship from the State Scholarship Foundation of Greece. He has been a graduate student in the Department of Electrical Engineering, University of Illinois, Urbana since fall 1977. In January 1979 he received the M.S. degree in electrical engineering. Since then he has been a Ph.D. candidate working on multi-time-scale modeling and its application to large scale systems such as power and economic systems. From August 1977 to June 1982 he has been a research assistant in the Decision and Control Laboratory of the Coordinated Science Laboratory.

Mr. Peponides is a member of I.E.E.E. and the honorary society of Phi Kappa Phi.